ETH
Eidgenössische Technische Hochschule Zürich
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Algorithms, Probability, and Computing **Exercises KW39** **HS17**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 13–15 CAB G 56

  Group B: Wed 13–15 CHN D 44

  Group C: Wed 16–18 CAB G 52

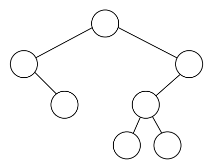
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on September 27, 2017. Please hand in your solutions in the lecture, no later than September 26.

The exercise sessions on September 27 will feature two additional in-class exercises.

Exercise 1: A Random Tree? How Random?

Determine the probability of the following tree with 7 nodes.



What is the smallest, what is the largest possible probability of a tree with 7 nodes?

Exercise 2: Very Deep Nodes

Let $n \in \mathbb{N}$. Show that the expected number of nodes of depth $n-1$ in a random search tree for n keys is $\frac{2^{n-1}}{n!}$. What is the probability that there is a node of depth $n-1$?

Exercise 3: High Trees

Let $n \in \mathbb{N}$. Determine the number of trees of height $n-2$ in $\mathcal{B}_{[n]}$.

(Recall that $\mathcal{B}_{[n]}$ is the set of all search trees with keys $1, 2, \ldots, n$.)

Exercise 4: Solving Recurrences

Determine closed forms for the following recursively defined series:

(1) For $n \in \mathbf{N}$,

$$a_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 1 + \frac{1}{n}\sum_{i=1}^{n-1} a_i, & \text{otherwise.} \end{cases}$$

(2) For $n \in \mathbf{N}$,

$$b_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 2 + \sum_{i=1}^{n-1} b_i, & \text{otherwise.} \end{cases}$$

(3) For $n \in \mathbf{N}_0$,

$$c_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ n - 1 + \sum_{i=1}^{n} \frac{c_{i-1} + c_{n-i}}{2}, & \text{otherwise.} \end{cases}$$

(4) For $n \in \mathbf{N}_0$,

$$d_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ 1 + 2\sum_{i=0}^{n-1} (-1)^{n-i} d_i, & \text{otherwise.} \end{cases}$$

(5) For $n \in \mathbf{N}_0$,

$$e_n = \begin{cases} 1, & \text{if } n = 0, \text{ and} \\ 1 + n\, e_{n-1}, & \text{otherwise.} \end{cases}$$

Exercise 5: Descendants of the Smallest Key

Let S_n denote the number of keys that are descendants of the smallest key. For example, in the tree below, $S_n = 5$, because the elements $1, 2, 3, 4, 6$ are descendants of 1.

Compute $\mathbf{E}[S_n]$.