ETH
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Algorithms, Probability, and Computing **Exercises KW40** **HS17**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 13–15 CAB G 56

  Group B: Wed 13–15 CHN D 44

  Group C: Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer”, then a formal proof is **always** required.

The following exercises will be discussed in the exercise class on October 4, 2017. Please hand in your solutions not later than October 3.

Exercise 1: Number of Leaves

Let $n \in \mathbf{N}$. Determine the expected number of leaves in a random search tree for n keys.

Exercise 2: Random Decline

Let $n \in \mathbf{N}$. We consider the following random process: First we choose a number $k_1 \in_{\text{u.a.r.}} [n]$, then a number $k_2 \in_{\text{u.a.r.}} [k_1 - 1]$, ... In general, we choose $k_{i+1} \in_{\text{u.a.r.}} [k_i - 1]$ until we have reached $k_N = 1$.

(1) Determine $\mathbf{E}[N]$ (in terms of n), i.e. the expected number of numbers chosen altogether.

(2) Determine $\mathbf{E}[k_1 + k_2 + \cdots + k_N]$.

Exercise 3: Maximum Expectation vs. Expected Maximum

Let $n \in \mathbf{N}$.

(1) Define random variables X_i with $\mathbf{E}[X_i] = O(1)$ for $i \in [n]$ and $\mathbf{E}[\max_{i=1}^{n} X_i] \geq n$.

(2) Define n mutually independent random variables X_i with $\mathbf{E}[X_i] = O(1)$ for $i \in [n]$ and $\mathbf{E}[\max_{i=1}^{n} X_i] \geq n$.

Exercise 4: Size of Subtrees

Let $i \in \mathbf{N}$, $n \in \mathbf{N}_0$, $i \leq n$. For a random search tree for n keys, let $W_n^{(i)}$ be the random variable for the number of nodes in the subtree rooted at the node of rank i (including the node itself).

(1) Show that $\mathbf{E}\left[\sum_{i=1}^{n} W_n^{(i)}\right] = n + \mathbf{E}\left[\sum_{i=1}^{n} D_n^{(i)}\right]$, where $D_n^{(i)}$ is the random variable for the depth of the node of rank i.

(2) Show that $\mathbf{E}\left[W_n^{(i)}\right] = 1 + \mathbf{E}\left[D_n^{(i)}\right]$ for all $i \in [n]$.

(3) Determine $\mathbf{E}\left[\max\{W_n^{(i)} : i \in [n]\}\right]$.

Exercise 5: Advanced Recurrences

Solve the following recurrence relations:

(a) Solve for $\{a_n\}_{n \in \mathbf{N}}$:

$$a_n = \begin{cases} 2 & \text{if } n = 1, \\ 4 \prod_{j=1}^{n-1} a_j & \text{if } n \geq 2. \end{cases}$$

(b) Solve for $\{b_n\}_{n \in \mathbf{N}_0}$:

$$b_n = \begin{cases} 7 & \text{if } n = 0, \\ 1 + 2 \sum_{j=1}^{n} (-1)^j b_{j-1} & \text{if } n \geq 1. \end{cases}$$

(c) Solve for $\{c_n\}_{n \in \mathbf{N}_0}$: (Hint: try to come up with a solution candidate, then prove it)

$$c_n = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ c_{n-1} + 2c_{n-2} - 2c_{n-3} & \text{if } n \geq 3. \end{cases}$$