Eidgenössische Technische Hochschule Zürich
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Algorithms, Probability, and Computing Exercises KW41 HS17

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A/B: Wed 13–15 CAB G 56

  Group C: Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise class on October 11, 2017. Please hand in your solutions not later than October 10.

Exercise 1: Building a Treap

Consider the process of inserting the keys $\{1, 2, \ldots, n\}$ into an empty treap in the order $(1, 2, \ldots, n)$.

(a) During this process, what is the expected number of changes of the root of the treap? (We also count the very first insertion as a change of the root.)

(b) For a given key i: What is the probability that i occurs as the right child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

(c) What is the expected number of elements that occur as the left child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

Exercise 2: Comparisons in Quicksort

Let $i, j, n \in \mathbf{N}$, $i < j \leq n$. What is the probability that the randomized procedure quicksort() applied to a set of n numbers compares the element of rank i with the element of rank j?

Exercise 3: Quickselect vs. Random Search Trees

Let $X_{k,n}$ be the random variable for the number of comparisons made by quickselect when searching for the element of rank k in a set of n numbers. Define a random variable on random search trees with the same distribution.

Exercise 4: Two Closest Numbers

Suppose you are given a finite set $S \subseteq \mathbf{R}$, $2 \leq |S|$, which is to be preprocessed so that for query $q \in \mathbf{R}$ the answer is 'the' set $\{b_1, b_2\} \subseteq S$ of the two closest numbers in S (i.e. $\max\{|b_1 - q|, |b_2 - q|\} \leq \min_{a \in S \setminus \{b_1, b_2\}} |a - q|$). Follow the locus approach for the problem and describe the resulting partition of regions of equal answers (and be aware of the ambiguity issue, i.e. the 'the' has to be taken with caution).

Exercise 5: Finding a Key vs. Line Hitting Convex Polygon

Given a sorted sequence $a_0 < a_1 < \cdots < a_{n-1}$ of n real numbers, we consider the convex polygon C with vertices $\left((a_i, a_i^2)\right)_{i=0}^{n-1}$. For $k \in \mathbf{R}$, show that the line with equation $y = 2kx - k^2$ intersects C iff $k \in \{a_0, a_1, \ldots, a_{n-1}\}$.

Remark: This exercise is supposed to exhibit that deciding whether a line intersects a convex polygon cannot be easier than deciding whether a query key is in a given set of keys.