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Algorithms, Probability, and Computing — Exercises KW51 — HS17

The following exercises will be discussed in the exercise class on December 20, 2016. Please hand in your solutions not later than December 19.

Exercise 1: Coloring neighbors in a bipartite graph

Let $G = (V \cup W, E)$ be a bipartite graph with bipartitions V and W and $|V| = |W| = n$. Assume that every vertex in V has degree at least $c \ln n$ for some sufficiently large constant c (20 and even less will suffice). We assume that every vertex knows whether it is in W or in V.

(a) Give a randomized constant round algorithm for 2-coloring the vertices in W so that every vertex in V has a neighbor of each color with high probability.

(b) Give a randomized constant round algorithm for 2-coloring the vertices in W so that for every $\epsilon > 0$ every vertex $v \in V$ is adjacent to at least $(\frac{1}{2} - \epsilon) \cdot \deg(v)$ many vertices of each color.

Can you find a deterministic $O(poly(\log n))$ round algorithm? This may be a hard question.

Exercise 2: Regularized Luby's algorithm

Consider a regularized variant of Luby's MIS algorithm, as follows: The algorithm consists of $\log \Delta + 1$ phases, each made of $200 \log n$ consecutive steps. Here Δ denotes the maximum degree in the graph. In each step of the i-th phase, each remaining node is marked with probability $\frac{2^i}{10\Delta}$. Different nodes are marked independently. Then marked nodes who do not have any marked neighbor are added to the MIS, and removed from graph along with their neighbors. If at any time a node v becomes isolated, thus none of its neighbors remain, then v is also added to the MIS and gets removed from the graph.

(a) Argue that the set of vertices added to the MIS is always an independent set.

(b) Prove that with high probability, by the end of the i-th phase, in the remaining graph each node has degree at most $\frac{\Delta}{2^i}$.

(c) Conclude that the set of vertices added to the MIS is a *maximal* independent set, with high probability.

Exercise 3: Locally minimal coloring

For a graph $G = (V, E)$, a proper coloring $\phi : V \to \{1, \ldots, Q\}$ is called *locally-minimal* if it is a proper coloring, meaning that no two adjacent vertices v and u have $\phi(v) = \phi(u)$, and moreover, for each node v colored with color $q = \phi(v) \in \{1, 2, \ldots, Q\}$, all colors 1 to $q - 1$ are used in the neighborhood of v. That is, for each $i \in \{1, \ldots, q - 1\}$, there exists a neighbor u of v such that $\phi(u) = i$.

(a) Assume we are given a $T(n, \Delta)$-round algorithm for computing a $(\Delta + 1)$-vertex coloring in any n-node graph with maximum degree Δ. Use this algorithm as a black box to compute a locally-minimal $(\Delta+1)$-coloring in $T(n, \Delta) + O(\Delta)$ rounds, in an n-node graph with maximum degree Δ.

Exercise 4: Reductions

Let $G = (V, E)$ be a graph maximum degree Δ. We have seen in the lecture that we can use an MIS algorithm to solve the $(\Delta + 1)$-coloring problem. In other words, we can reduce $(\Delta + 1)$-coloring to the MIS problem. Find a reduction from the problem of finding a maximal matching to the MIS problem and a reduction from the problem of finding a $(2\Delta - 1)$-edge coloring to the MIS problem. Recall that a c-edge coloring of G is a function $\phi : E \to \{1, \ldots, c\}$ such that no two neighboring edges are mapped to the same color/number.

If you have an algorithm for MIS that runs in $T(n, \Delta)$ rounds, what is the runtime you get for the other problems via the reduction?