ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, David Steurer, Emo Welzl, Peter Widmayer

Solution 1: Reducing the Number of Colors in a Single Round

To be precise, the method reduces the number of colors from k to $k' = 2\lceil \log_2 k \rceil$. (We can set $C_0 = 2$, i. e. we do not really need the constant C_0 – except that we should take note of the fact that $k' < k$ holds only for $k \geq 7$. For some small values it can even happen that the number of colors increases.)

Let $\phi_{\text{old}} : V \to \{0, \ldots, k-1\}$ be given.

Well-definedness: Let $v \in V$ and let u be its parent. Since ϕ_{old} is a proper coloring, we have $\phi_{\text{old}}(v) \neq \phi_{\text{old}}(u)$ and hence there actually exists a smallest index i_v wherein these two numbers differ. If we let $\ell = \lfloor \log_2(k-1) \rfloor + 1 = \lceil \log_2 k \rceil$ denote the number of bits in the bit representation of the largest color $(k-1)$, then we have $i_v \in \{1, \ldots, \ell\}$. Hence we have defined a map

$$\begin{array}{rcrcl} \phi_{\text{new}} & : & V & \longrightarrow & [\ell] \times \{0,1\} \\ & & v & \longmapsto & (i_v, b_v) \end{array}$$

whose range is indeed of cardinality at most $2\ell = k'$.

Properness: Assume that v and w are neighbours such that $\phi_{\text{new}}(v) = \phi_{\text{new}}(w)$. Assume without loss of generality that v is the parent of w, and let u denote the parent of v. Also without loss of generality assume that $b_v = 0$. Then the i_vth bit of $\phi_{\text{old}}(u)$ is 1, and from $i_v = i_w$ it follows that b_w is the i_vth bit of w and $b_w = 1$. This contradicts the assumption $(b_v = b_w)$.

Solution 2: 7-Coloring Planar graphs

Let $G = (V, E)$ be the planar graph we are interested in with $n = |V|$.

(a) A planar graph has at most $3n - 6$ vertices and therefore $\sum_{v \in V} \deg(v) \leq 6n$. If the number of vertices of degree at least 7 was more than $\frac{6}{7}n$, then it would hold that $\sum_{v \in V} \deg(v) > 7 \cdot \frac{6}{7}n = 6n$, a contradiction.

(b) We do the same peeling process as in the proof of Theorem 8.14. We partition the vertex set into sets $L_1, \ldots, L_t$ for some t similarly as we did in the proof of Theorem 8.14. Specifically, for every $i = 1, \ldots, t$ we consider the graph that remains after removing all vertices in $L_1, \ldots, L_{i-1}$ from G and we let L_i be the set of vertices with degree at most 6 in this graph. Because every subgraph of a planar graph is planar and because there is always at least $\frac{1}{7}$-fraction of vertices with degree at most 6, we have that $t \in O(\log n)$. We get the orientation asked in the exercise description by orienting the edges when they get removed (due to removing a vertex). Every edge that is to be removed we orient it edge away from the vertex removed and we break ties arbitrarily. Because of the degree property this guarantees that every vertex will have at most 6 outgoing edges.

(c) Each vertex labels one of its ≤ 6 outgoing edges with a label $\{1, \ldots, 6\}$. Now we have 6 labeled graphs and each of the labeled graphs has at most one outgoing edge per vertex. As is mentioned in the lecture, the $O(\log^* n)$ algorithm for coloring rooted trees also works in this setting so we can 3-color each of these 6 labeled graphs (in parallel) in $O(\log^* n)$ rounds. By considering the Cartesian products of the colorings we get a 3^6-coloring.

(d) For avoiding confusion, let us consider that the 3^6-coloring we constructed in (c) uses the colors $\{1, \ldots, 3^6\}$ and the new 7-coloring uses the colors $\{a, b, c, d, e, f, g\}$. We consider $t = O(\log n)$ phases, where each phase consists of 3^6 rounds. Starting with an initially empty coloring we will maintain the invariant that after the ith phase, $i = 0, \ldots, t$ we have properly 7-colored all vertices in $L_t, \ldots, L_{t-i+1}$. This invariant holds initially so we consider properly coloring all vertices in L_{t-i} given a proper 7-coloring of the vertices in $L_t, \ldots, L_{t-i+1}$. For that we consider the 3^6 coloring from (c) which we think of as a scheduling so that a vertex from $v \in L_{t-i}$ that was assigned the color $c \in \{1, \ldots, 3^6\}$ will choose its final color from $\{a, \ldots, g\}$ on the c'th round of the $(i+1)$st phase. Say that a vertex $v \in L_{t-i}$ was given the color c in (c) and it is now the round for v to choose its color. Then by construction v has at most 6 edges adjacent to vertices in $L_t, \ldots, L_{t-i}$. Vertex v can also not be adjacent to any other vertex in L_{t-i} with color c that is choosing its color at the same round. Therefore vertex v has one free color it can choose that will produce a proper coloring.

(e) Consider a graph consisting of a clique with roughly $\sqrt{n}$ vertices and a matching with roughly $2(n - \sqrt{n})$ vertices. Then the number of edges is upper bounded by $3n$ but for every orientation of this graph there will be a vertex with $\Theta(\sqrt{n})$ outgoing edges.