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Algorithms, Probability, and Computing **Solutions KW 51** **HS17**

Solution 1: Coloring neighbors in a bipartite graph

For both (a) and (b) the algorithm is for every vertex in W to simply pick a random color from $\{0, 1\}$ independently and to select that color.

(a) The probability that a single node $v \in V$ has a monotonically colored neighborhood is at most $2 \cdot \left(\frac{1}{2}\right)^{c \log n} = \frac{2}{n^c}$. By union bound the probability that no vertex has a monotonically colored neighborhood is at most $\frac{2n}{n^c} = O(\frac{1}{n^{c-1}})$. Requiring that $c \geq 3$, say, gives the desired high probability guarantee.

(b) Now we want something stronger, namely that the number of colors that every vertex $v \in V$ sees in its neighborhood is close to the expected value. For a fixed vertex v let $X_1, \ldots, X_d \in \{0, 1\}$ denote the colors of the vertices in W that are adjacent to v. Here $d = \deg(v)$. Let $X = \sum_{i=1}^{d} X_i$ be the number of vertices adjacent to v colored with color 1. We have that $\mu := \mathbb{E}[X] = \frac{d}{2} \geq \frac{c}{2} \log n \geq \frac{c}{4} \ln n$ by the assumption in the exercise description. Then by the Chernoff bound (Theorem 8.50 from the lecture notes) we get that

$$\Pr[X \leq (1 - 2\epsilon)\frac{d}{2}] \leq 2e^{-4\epsilon^2\mu/3} \leq 2e^{-\epsilon^2\mu} \leq 2e^{-\epsilon^2 c/4 \ln n} = 2n^{-c\epsilon^2/4} \leq \frac{2}{n^3}$$

where the last inequality holds if $c \geq \frac{12}{\epsilon^2}$. Union bounding over all the vertices gives us the required guarantee with high probability (the failure probability is $O(\frac{1}{n^2})$).

It is an open question to find a deterministic polylogarithmic round algorithm for either of the coloring problems. Many problems in the LOCAL model could be solved faster deterministically if we knew such an algorithm.

Solution 2: Regularized Luby's algorithm

(a) We prove by induction over the steps of the algorithm that the set of vertices added to the MIS form an independent set and that no remaining vertex is adjacent to the vertices in the so far constructed MIS. We start with the empty set so the base case is trivially true. Assume now that the statement holds after some number of steps of the algorithm. In the next step the vertices added to the MIS form an independent set in the remaining graph since no two neighboring vertices can ever be added. Moreover, the union of the previous MIS and the added vertices is an independent set in the original graph since none of the vertices in the previous MIS were adjacent to the vertices added. Because we remove the vertices adjacent to the added vertices the second condition of the induction is also satisfied which finishes the proof.

(b) We prove the statement by induction over the phases. The initial degree is at most $\Delta = \frac{\Delta}{2^0}$ so the statement holds in the beginning. Assume now that at the end of the i-th phase the remaining graph G' has maximum degree at most $\frac{\Delta}{2^i}$, with high probability and consider the $(i+1)$-st phase. We want to show that with high probability in the next $200 \log n$ steps the maximum degree will be at most $\frac{\Delta}{2^{i+1}}$.

Fix an arbitrary vertex $v \in G'$ of degree at least $\frac{\Delta}{2^{i+1}}$ (and at most $\frac{\Delta}{2^i}$). In every step of the current phase, as long as v has degree at least $\frac{\Delta}{2^{i+1}}$, the probability that some neighbor of v gets marked is at least

$$1 - \left(1 - \frac{2^{i+1}}{10\Delta}\right)^{\frac{\Delta}{2^{i+1}}} \geq 1 - e^{-1/10} \geq 0.05.$$

Let $u \in \delta(v)$ be a neighbor of v and consider the conditional probability $P := \Pr[\text{no vertex in } \delta(u) \text{ is marked} | u \text{ is marked}]$, i.e., the probability that u gets added to the MIS which results in the removal of v from the graph. Because $\deg(u) \leq \frac{\Delta}{2^i}$ we have that

$$P \geq \left(1 - \frac{2^{i+1}}{10\Delta}\right)^{\frac{\Delta}{2^i}} \geq 4^{-\frac{2^{i+1}}{10\Delta} \cdot \frac{\Delta}{2^i}} = 4^{-1/5} \geq 0.75.$$

Therefore

$$\Pr[v \text{ is removed in one step}] \geq 0.05 \cdot 0.75 \geq 1/50.$$

The probability of v not being removed in $200 \log n$ steps is at most

$$\left(1 - \frac{1}{50}\right)^{200 \log n} \leq 2^{-\frac{200 \log n}{50}} = n^{-4}.$$

By a union bound, the probability that no node has degree more than $\frac{\Delta}{2^{i+1}}$ after the $(i+1)$-st phase is at most $1 - \frac{1}{n^3}$.

(c) Since we always remove vertices that either join the MIS or that are neighbors of vertices that joined the MIS, the output is a maximal independent set if we can show that after $\log \Delta + 1$ phases there are no vertices in the graph with high probability.

By (b) the degree of every vertex at the end of phase $\log \Delta + 1$ is at most $\frac{\Delta}{2\Delta} = 1/2$ with high probability. Because degrees are always integers, the degrees have to be 0. Since we add isolated vertices to the MIS, there can be no vertices in the graph. This concludes the proof.

Remark. Of course since we get the degree reduction in every phase with probability at least $1 - n^{-3}$, the probability that all phases succeed is at least $1 - n^{-3} \log n$ by union bound which is still with high probability.

Solution 3: Locally minimal coloring

(a) We first use $T(n, \Delta)$ time to construct a $(\Delta+1)$-coloring which we then modify in $O(\Delta)$ rounds to a locally minimal $(\Delta+1)$-coloring. Consider the following algorithm that uses the initial $(\Delta+1)$-coloring as a schedule, so that vertices will choose new colors in the order $1, 2, \dots, \Delta+1$. If it is turn for a vertex with color i to act, we let it pick a new color that is the lowest color that does not exist in its neighborhood.

Notice that this retains a proper coloring after every recoloring round since two vertices that recolor themselves in the same round are never adjacent and they always pick a color that causes no conflict locally. The coloring is also a locally-minimal coloring because the new color a vertex receives satisfies the locally-minimal coloring condition and this condition stays satisfied for the remainder of the algorithm since the new color of the vertex is always at most its current color.

Solution 4: Reductions

The reductions and the resulting runtimes are illustrated in Figure 1. The reduction from $(\Delta+1)$=vertex coloring to MIS that we have seen in the lecture worked by constructing the graph $G \times K_\Delta$ which has Δn vertices and maximum degree $\Delta \cdot \binom{\Delta}{2} = O(\Delta^3)$. Therefore we can solve $(\Delta+1)$=vertex coloring in time $T(\Delta n, O(\Delta^3))$.

The other two reductions work via the *line graph*. Given a graph $G = (V, E)$, its line graph is $L(G) = (E, F)$ where F consists of all edge pairs that are adjacent in G. Notice that a maximal independent set in the line graph corresponds to a maximal matching in the original graph. If G has maximum degree Δ, then its line graph has maximum degree at most $2\Delta-2$ and at most $O(n^2)$ vertices. This means that we can solve maximal matching in time $T(O(n^2), 2\Delta-2)$. Similarly, a vertex coloring of the line graph corresponds to an edge coloring of the original graph. The runtime we get in the end is $T(O(\Delta^2 n^2), O(\Delta))$.

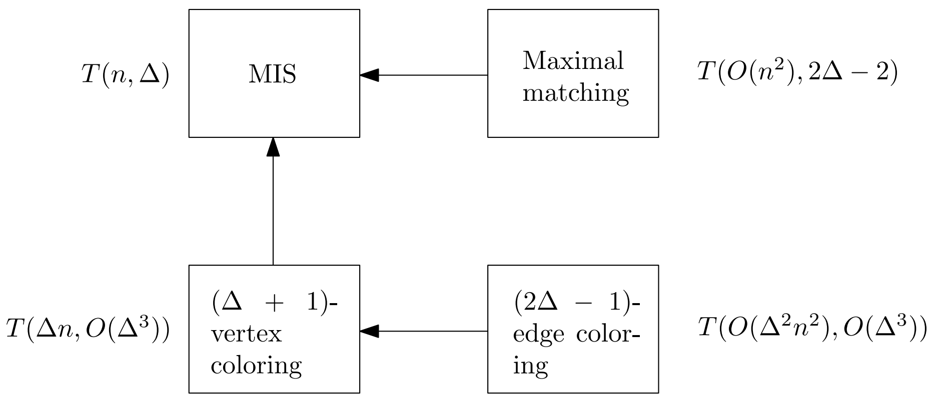

Figure 1: The reductions and the resulting runtimes in terms of the function $T(\cdot, \cdot)$.