ETH
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Algorithms, Probability, and Computing **Exercises KW43** **HS18**

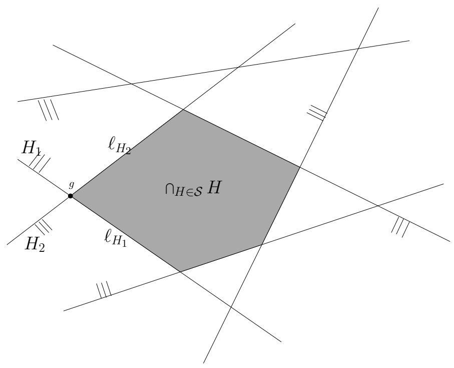
The following exercises will be discussed in the exercise class on October 24, 2018. These are "in-class" exercises, which means that we do not expect you to solve them before the exercise session. Instead, your teaching assistant will solve them with you in class.

Exercise 1

Show that the general position assumption in trapezoidal decomposition can be relaxed to allow in particular vertical segments and segments that intersect in their endpoints. We still assume that the segments are non-crossing. What if we in addition allow a query point to lie on a segment or to share an x-coordinate with some endpoint?

Exercise 2

Suppose we are given a set $\mathcal{S}$ of n closed halfspaces in the plane. For each $H \in \mathcal{S}$, let $\ell_H \subset H$ denote its boundary line. We assume that the halfspaces are in general position such that no two boundary lines are parallel and no three boundary lines meet in a single point. Consider the input to be given in the form of linear inequalities, say.



In this task we are interested in a randomized algorithm to decide whether the intersection of the given halfspaces is non-empty, that is whether $R(\mathcal{S}) = \emptyset$ for $R(\mathcal{S}) := \bigcap_{H\in\mathcal{S}} H$, or not. If $\mathcal{S}$ has a non-empty intersection, we would also be interested in a *certificate point*, that is in a point $x \in \bigcap_{H\in\mathcal{S}} H$ to demonstrate non-emptiness. To make your calculations simpler, we want to make certificate points unique. To this end, we assume $|\mathcal{S}| \geq 2$ and fix, arbitrarily, two halfspaces $H_1, H_2, \in \mathcal{S}$. The region $R(\mathcal{S})$ is obviously contained in a wedge formed by the lines ℓ_{H_1} and ℓ_{H_2} (see figure). Before starting any algorithm, you may assume that the input is rotated[1] first in such a way that this wedge opens to the right and the intersection point $g \in \ell_{H_1} \cap \ell_{H_2}$ acts as a guard that no point in $R(\mathcal{S})$ can have a smaller x-coordinate than g (see figure). We then define for any $\mathcal{S}' \subseteq \mathcal{S}$ with $H_1, H_2 \in \mathcal{S}'$ the unique certificate point $c(\mathcal{S}')$ as the point in $R(\mathcal{S}')$ that has the smallest x-coordinate. You may assume that H_1 and H_2 are fixed before and known to all your algorithms below.

Following are your tasks:

(a) Let $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above) and let $H \in \mathcal{S} \setminus \{H_1, H_2\}$ be an arbitrary one of the halfspaces. Prove: if $R(\mathcal{S}) \neq \emptyset$, then either $c(\mathcal{S}) = c(\mathcal{S} \setminus \{H\})$ or $c(\mathcal{S}) \in \ell_H$.

(b) Let $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above) and let $H \in \mathcal{S} \setminus \{H_1, H_2\}$ be an arbitrary one of the halfspaces. Assume that $R(\mathcal{S} \setminus \{H\}) \neq \emptyset$. Write down a deterministic algorithm that runs in time linear in $n = |\mathcal{S}|$ and that on input $(\mathcal{S}, H, c(\mathcal{S} \setminus \{H\}))$ determines whether $R(\mathcal{S}) \neq \emptyset$ and if so outputs $c(\mathcal{S})$.

(c) Let again $|\mathcal{S}| \geq 3$ (with H_1 and H_2 as described above). Using (b), write down a randomized algorithm which, given $\mathcal{S}$, determines whether $R(\mathcal{S}) \neq \emptyset$ and if so outputs $c(\mathcal{S})$. Your algorithm should run in expected time linear in $n = |\mathcal{S}|$.

Exercise 3

Show that every linear program can also be converted into the following *equational form*:

$$\text{maximize } \mathbf{c}^T \mathsf{x} \text{ subject to } A\mathsf{x} = \mathbf{b},\ \mathsf{x} \geq \mathbf{0}.$$

What is the maximum increase in the number of variables and in the number of constraints in such a transformation?

[1] this rotation can always be done such that we also do not have vertical or horizontal lines, which you may assume