Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, David Steurer, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing **Special Assignment 1** **HS18**

- The solution is due on **Monday, October 29, 2018** by **2:15 pm**. Please bring a print-out of your solution with you to the lecture. If you cannot attend (and please only then), you may alternatively email your solution as a PDF, likewise **until 2:15 pm**. We will send out a confirmation that we have received your file. Make sure you receive this confirmation within the day of the due date, otherwise complain timely.
- Write an exposition of your solution using a computer, where we strongly recommend to use LaTeX. **We do not grade hand-written solutions**.
- For geometric drawings that can easily be integrated into LaTeX documents, we recommend the drawing editor IPE, retrievable at `http://ipe7.sourceforge.net/` in source code and as an executable for Windows.
- Write short, simple, and precise sentences.
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer" or "justify intuitively", then a formal proof is **always** required. You can of course refer in your solutions to the lecture notes and to the exercises, if a result you need has already been proved there.
- We would like to stress that the ETH Disciplinary Code applies to this special assignment as it constitutes part of your final grade. The only exception we make to the Code is that we encourage you to verbally discuss the tasks with your colleagues. It is strictly prohibited to share any (hand)written or electronic (partial) solutions with any of your colleagues. We are obligated to inform the Rector of any violations of the Code.
- There will be two special assignments this semester. Both of them will be graded and the average grade will contribute 20% to your final grade.
- As with all exercises, the material of the special assignments is relevant for the (midterm and final) exams.

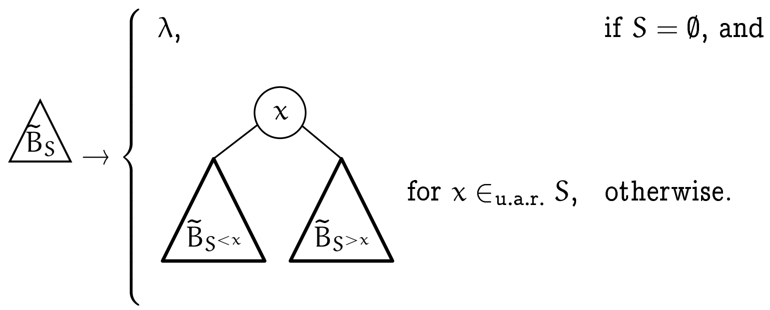
Exercise 1 20 points

(a) Let G be a multigraph with $n \geq 4$ nodes. We define an *almost minimum cut* in G to be a cut whose size is within a multiplicative factor 2 of the size of a minimum cut in G. Consider a modified version of the algorithm BasicMinCut, that contracts uniformly random edges of G until there are exactly 4 nodes remaining; then, it picks a cut uniformly at random among all cuts in the remaining graph and returns its size. Let K be the size of some almost minimum cut in G. Prove that the probability of this modified algorithm returning K is bounded by $\Omega(\frac{1}{n^4})$.

(b) For a graph $G = (V, E)$ and two distinguished nodes $s, t \in V$, an s-t cut is a partition of V into A and B such that $s \in A$ and $t \in B$. The size of such a cut is the number of edges between A and B. We seek an s-t cut of minimum size. Assume we apply BasicMinCut(G) to G; as the algorithm proceeds, the node s may get merged into a new node as the result of an edge being contracted; we call this node the s-node (initially s itself). Similarly, we have a t-node. As we run the algorithm, we ensure that we never contract an edge between the s-node and the t-node (this guarantees that in the end, we get an s-t cut). Prove that there are (multi-)graphs in which the probability that this algorithm finds a minimum s-t cut is exponentially small, i.e. $\mathcal{O}(1/c^n)$ for some constant $c > 1$.

Exercise 2 20 points

Recall that for a given set S of $n \in \mathbb{N}_0$ distinct real numbers, we defined random search trees recursively as follows



Assume that instead of choosing x uniformly at random (i.e., $x \in_{\text{u.a.r.}} S$), we choose $x \in S$ with probability $x/(\sum_{y \in S} y)$. Furthermore, let set S be equal to $[n] = \{1, \cdots, n\}$.

(a) For $i, j \in [n]$ and $j < i$, let the indicator random variable A_i^j be 1 if and only if node j is ancestor of node i. Prove that

$$\Pr[A_i^j = 1] = \frac{2j}{(i-j+1)(i+j)}.$$

(b) Determine the expected depth of the largest key.

Exercise 3 — 20 points

Let S, a set of n non-crossing segments in general position, be given. In the history graph of the trapezoidal decomposition built in random order, the segment above any fixed query point q can be found in expected $\mathcal{O}(\log n)$ time (cf. Theorem 3.11 in the lecture notes).

(a) Let q be a fixed query point, and suppose that the segments of S are inserted in random order $s_1, s_2, \dots s_n$ (u.a.r. from all permutations) while maintaining a trapezodial decomposition. Let the indicator random variable X_i be 1 if and only if the trapezoid containing q changes when s_i is added. Provide an example with three segments where the variables X_i are not mutually independent.

(b) Define indicator random variables $Y_i \geq X_i$ such that the variables Y_i are mutually independent and $\mathbf{E}[Y_i] = \min\{1, \frac{4}{i}\}$.

(c) Let $Y := \sum_{i=1}^{n} Y_i$. Obviously Y is an upper bound on the number of nodes of the history graph we need to traverse to locate q. Prove that $\mathbf{E}\left[e^{\ln(5/4)Y}\right] \leq n + 1$ and $\Pr\left[Y \geq \lambda \ln(n+1)\right] \leq \left(\frac{1}{n+1}\right)^{\lambda \ln(5/4) - 1}$ for $\lambda \in \mathbb{R}^{+}$.

(d) Prove that with probability at least $1 - \frac{2}{n+1}$, the query time is $O(\log n)$ in the *worst case* (i.e. for all possible query points q *at the same time*).
HINT: In which case do two points q and q' behave identically in *all* possible history graphs? How many such equivalence classes are there? Set λ accordingly and use the union bound.