linear programming

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Algorithms, Computing, and Probability, Fall 2018, ETH Zürich

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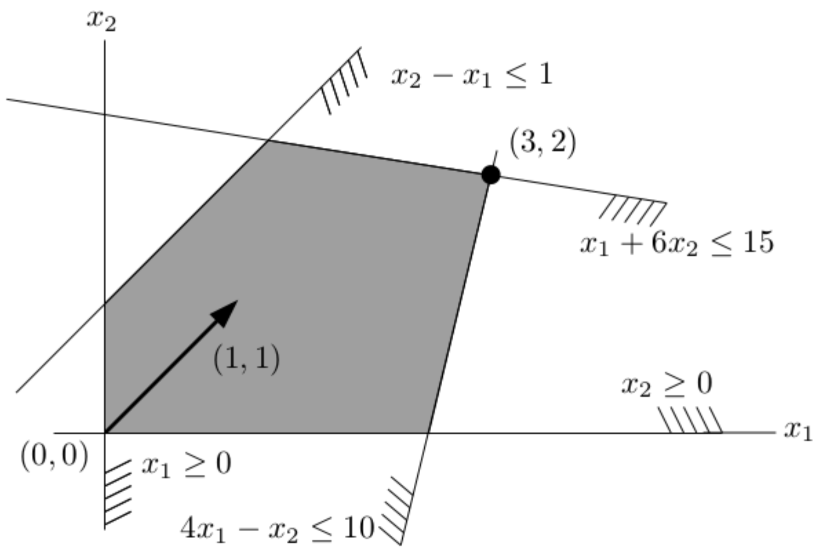
$$\begin{array}{ll} \text{Maximize} & x_1 + x_2 \\ \text{subject to} & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_2 - x_1 \leq 1 \\ & x_1 + 6x_2 \leq 15 \\ & 4x_1 - x_2 \leq 10. \end{array}$$

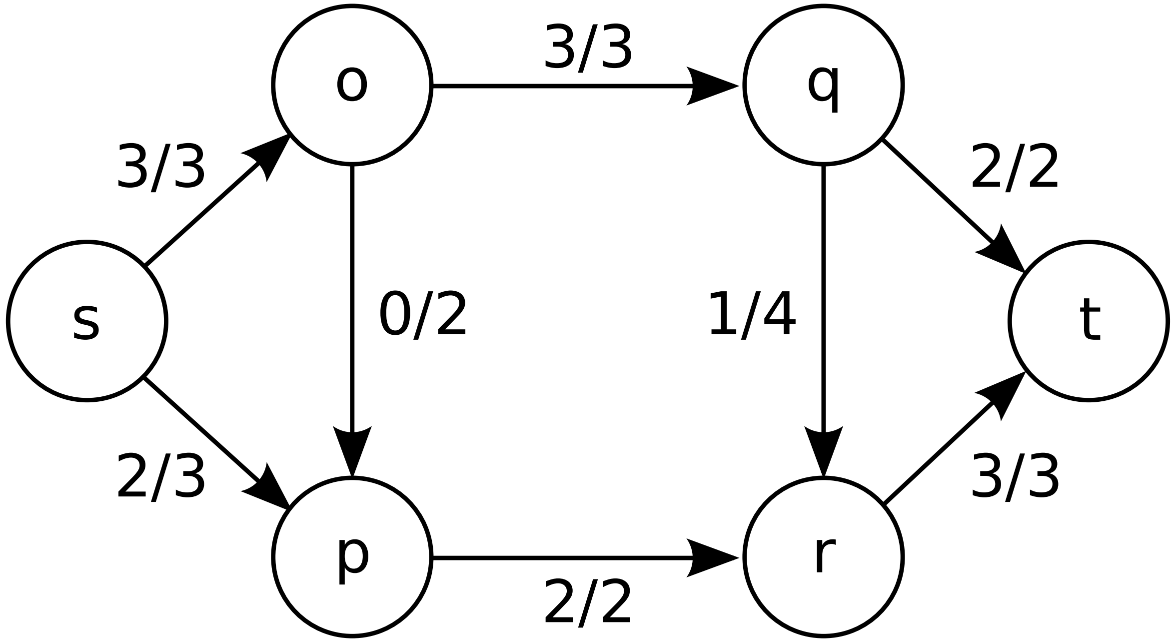


application: network flow

network flow

given: dir. graph $G = (V, E)$ with edge cap. $c\colon E \to \mathbb{R}_{\geqslant 0}$, source $s \in V$ and target $t \in V$

find: maximum flow from s to t in G within capacities c



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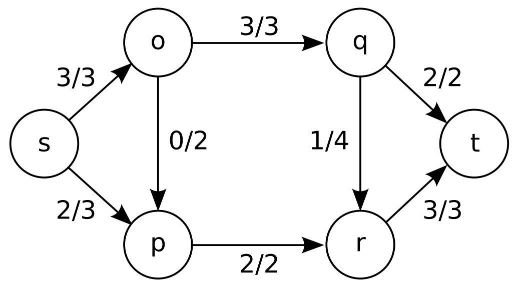
variables: $f_e \geqslant 0$ for all $e \in E$



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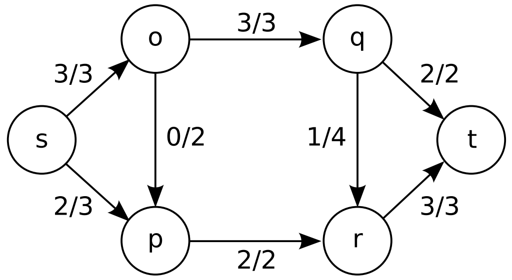

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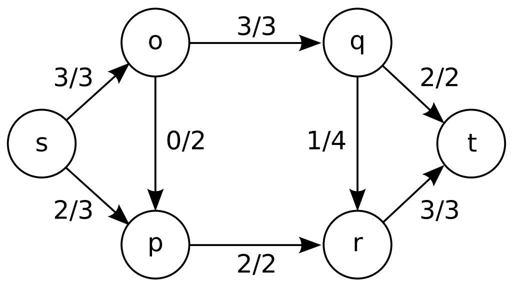

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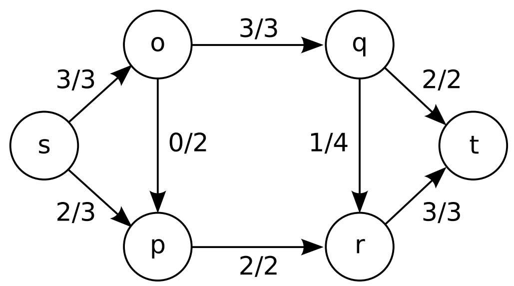

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objective: $\sum_{e \in \delta^+_G(s)} f_e - \sum_{e \in \delta^-_G(s)} f_e$ (net flow out of source)

geometry of linear programs

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(feasible region of linear program)

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corollary: *set of optima* of linear program is convex

basic feasible solution

consider linear program with constraints $\{Ax \leqslant b\}$

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examples

- traveling salesman problem
- network design (e.g., Steiner tree)
- edge-disjoint paths
- compressed sensing
- nonnegative matrix factorization

…

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typically, most of the "analysis work" is in step 3 and most of the "algorithmic work" is in step 2

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LP in P: there is a *polynomial-time algorithm* to decide the satisfiability of systems of linear inequalities

bounds on LP solutions

(or LP in NP)

notation: $\langle X \rangle =$ (binary) encoding size of object X

theorem: for every bounded linear program $\mathcal{L}$, there exists an optimal solution x^* such that $\langle x^* \rangle \leqslant \langle \mathcal{L} \rangle^{O(1)}$

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in the next slides, we will bound the encoding size of basic feasible solutions

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$$\begin{aligned} |\det A| &= \left|\textstyle\sum_{\pi\in S_n} \operatorname{sign}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}\right| \\ &\leqslant n! \cdot 2^{\langle A \rangle} = 2^{\langle A \rangle + O(n\log n)} \leqslant 2^{O(\langle A \rangle)} \end{aligned}$$

□

lemma: every basic feasible solution $\tilde{x}$ of $\mathcal{L}$ satisfies

$$\forall j \in [n].\ \langle \tilde{x}_j \rangle \leqslant O(\langle A \rangle + \langle b \rangle + n^2)$$

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proof: since $\tilde{x}$ is a BFS, it is the solution of a non-singular linear system $\{\tilde{A}x = \tilde{b}\}$, where each equations is either one of the equations in $\{Ax = b\}$ or of the form $x_i = 0$

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lemma: every basic feasible solution $\tilde{x}$ of $\mathcal{L}$ satisfies
$$\forall j \in [n].\ \langle \tilde{x}_j \rangle \leqslant O(\langle A \rangle + \langle b \rangle + n^2)$$

proof: since $\tilde{x}$ is a BFS, it is the solution of a non-singular linear system $\{\tilde{A}x = \tilde{b}\}$, where each equations is either one of the equations in $\{Ax = b\}$ or of the form $x_i = 0$

by *Cramer's rule*, $\tilde{x}_j = \det \tilde{A}_j / \det \tilde{A}$, where $\tilde{A}_j$ is obtained from $\tilde{A}$ by replacing the j-th column with $\tilde{b}$

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duality and Farkas lemma

(or LP in coNP)

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how could we certify that a system of linear inequalities is unsatisfiable?

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the theorem says that $\{Ax = b\}$ is unsatisfiable if and only if we can derive in this way the syntactically unsatisfiable equation $\{0^\top x = 1\}$

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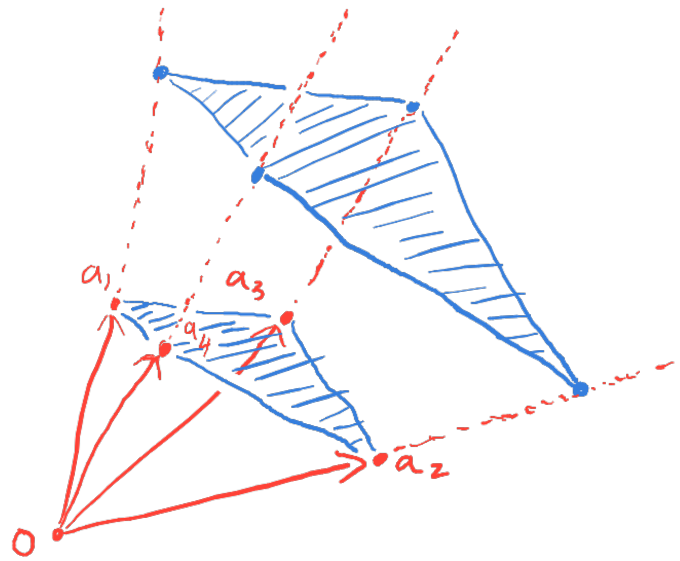
complexity interpretation: LP in coNP

geometry interpretation

columns $a_1, \ldots, a_n$ of matrix A generate *cone*

$$C = \{x_1 \cdot a_1 + \cdots + x_n \cdot a_n \mid x_1, \ldots, x_n \geqslant 0\}$$



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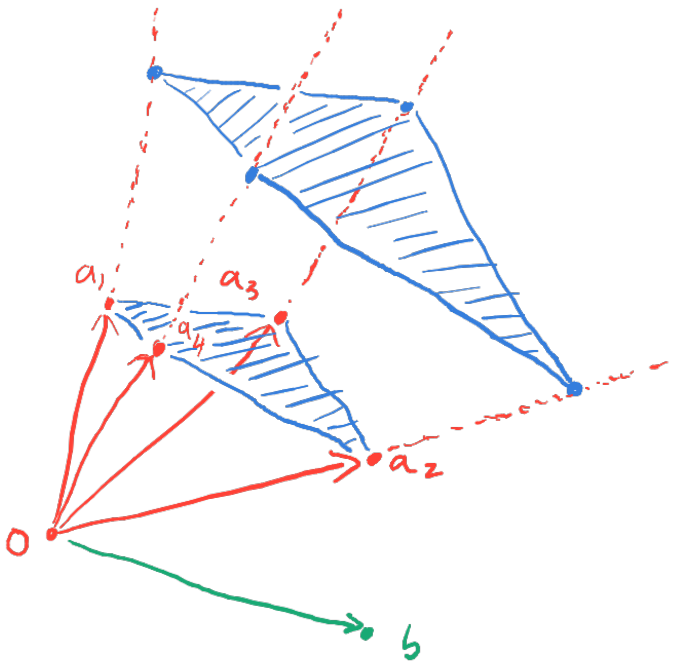

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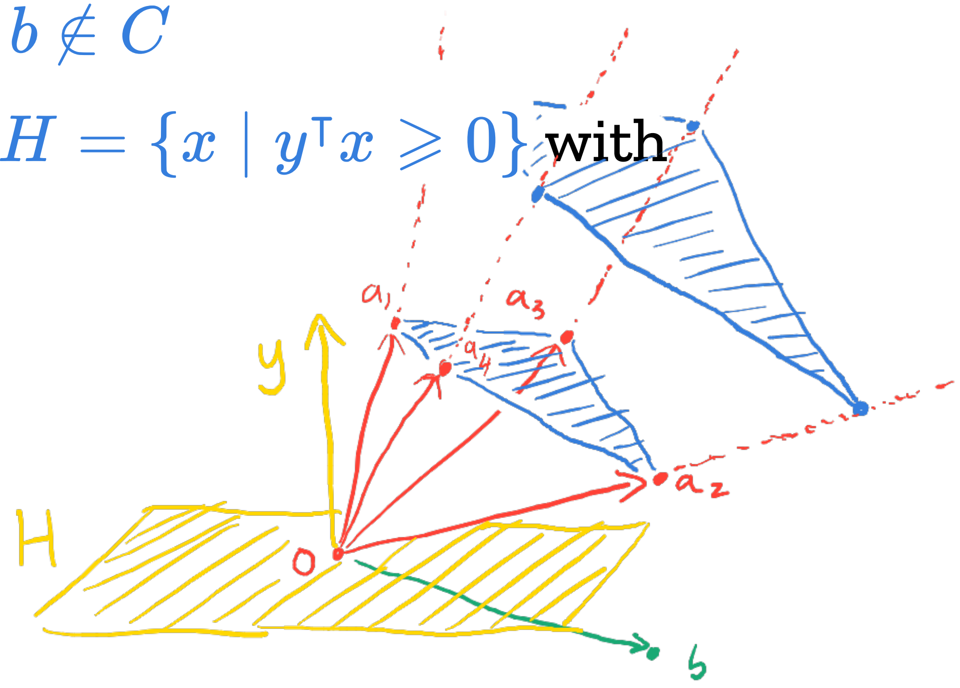

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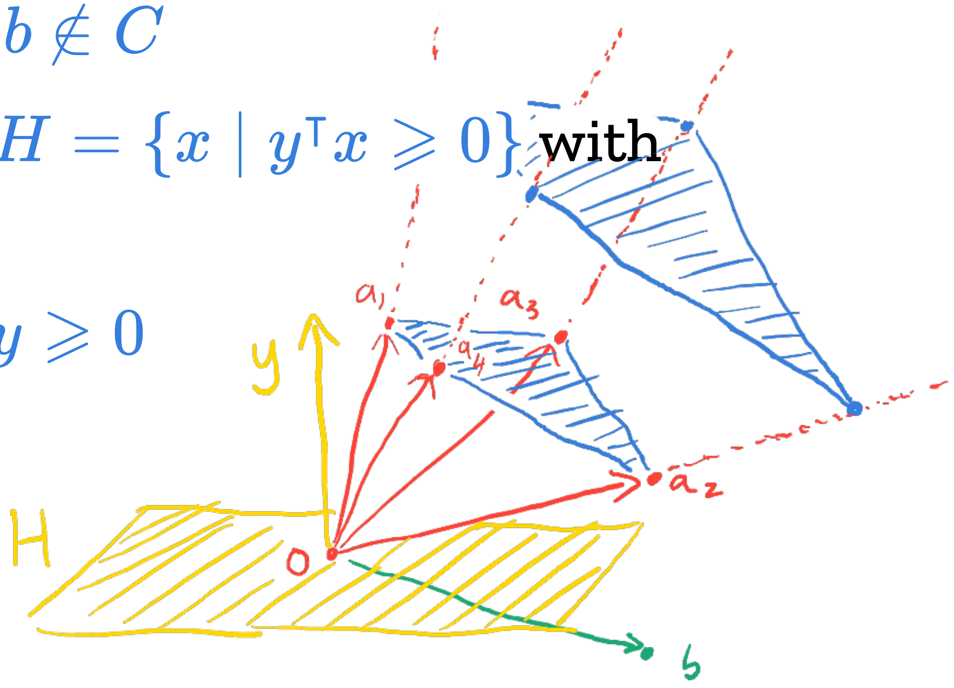

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weak-duality: optimal value of (P) $\leqslant$ optimal value of (D) (holds by construction)

consider solution y to (D)

$\forall$ solutions x to (P).

$$c^T x \leq (A^T y)^T x = y^T A x \leq y^T b = b^T y$$

$x \geq 0,\ A^T y \geq c$

$$\left((A^T y)^T x - c^T x = (A^T y - c)^T x \geq 0 \right)$$

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thus, $z > 0$; by rescaling y, we can assume $z = 1$; now y has objective value $< \gamma + \varepsilon$ for (D) □

Ellipsoid method

(or LP in P)


"All the News That's Fit to Print"

The New York Times

LATE CITY EDITION

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STUDENTS WARN U.S.

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A Soviet Discovery Rocks World of Mathematics

INSIDE

CUNY Finds Tuition Harm

Khachiyan's achievement received an *attention in the nonscientific press that is—to our knowledge—unpreceded in mathematics*. Newspapers and journals like The Guardian, Der Spiegel, Nieuwe Rotterdamsche Courant, Nepszabadsag, The Daily Yomiuri wrote about the *"major breakthrough in the solution of real-world problems".* The ellipsoid method even jumped on the *front page of The New York Times: "A Soviet Discovery Rocks World of Mathematics"* (November 7, 1979). Much of the excitement of the journalists was, however, due to *exaggerations and misinterpretations* - see LAWLER (1980) for an account of the treatment of the implications of the ellipsoid method in the public press.

(from textbook by Grötschel, Lovasz, and Schrijver)

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relaxed feasibility problem

given: $R > \varepsilon > 0$, polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

promise: $\exists y \in \mathbb{R}^n.\ \mathrm{Ball}(y, \varepsilon) \subseteq P \subseteq \mathrm{Ball}(0, R)$

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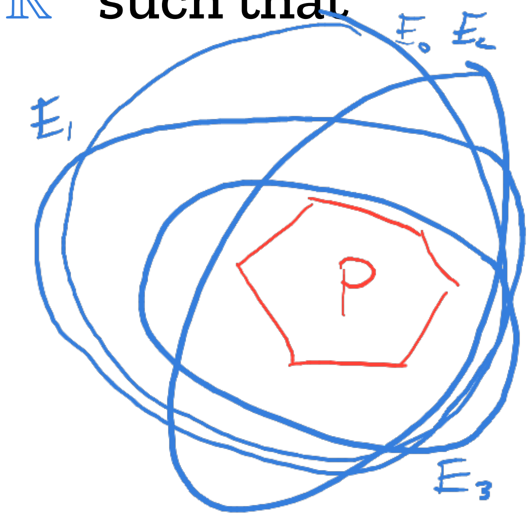

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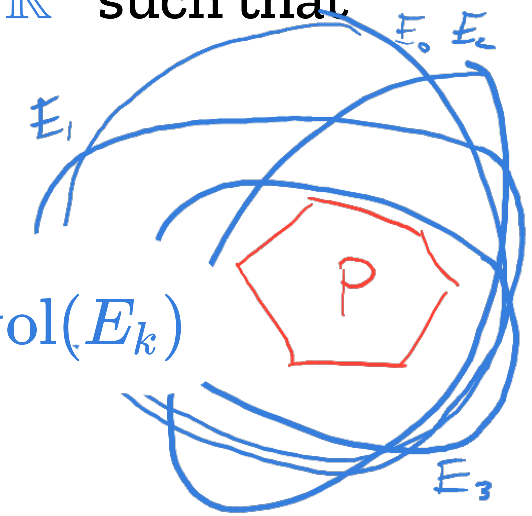
key property: alg. ensures $\mathrm{vol}(E_{k+1}) \leqslant e^{-1/n} \cdot \mathrm{vol}(E_k)$



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bird's eye view of ellipsoid method

compute *successively tighter outer-approximations for* P

concretely, sequence of ellipsoids $E_0, \ldots, E_N \subseteq \mathbb{R}^n$ such that

$$E_0, E_1, \cdots, E_N \supseteq P$$

where $E_0 = \mathrm{Ball}(0, R)$ and center of E_N is in P

key property: alg. ensures $\mathrm{vol}(E_{k+1}) \leqslant e^{-1/n} \cdot \mathrm{vol}(E_k)$

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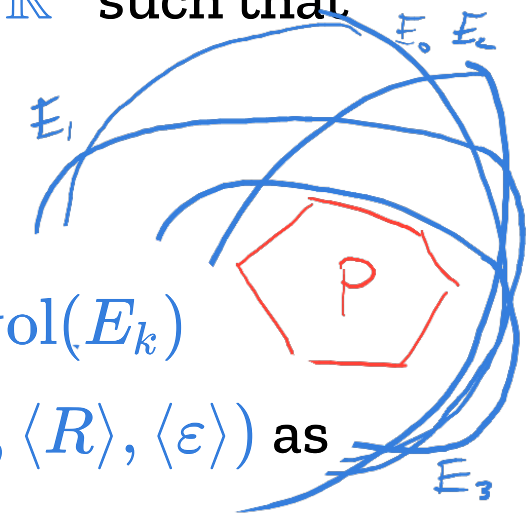

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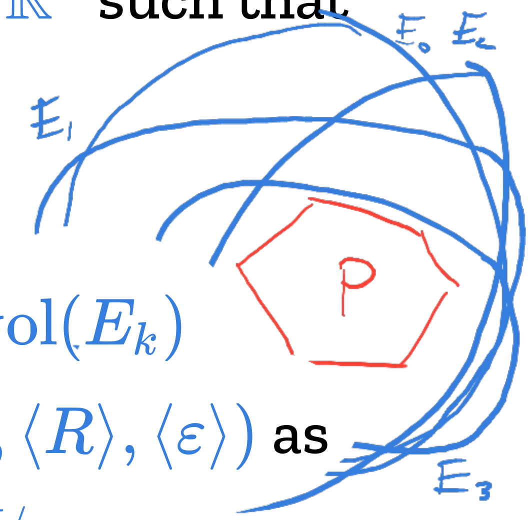

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also remains to prove (but will skip): no numerical issues

geometry of ellpsoids

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any non-singular affine transformation $x \mapsto Mx + s$ of

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strategy for proof of theorem:

- first prove unit-ball case $E = E_0$
- then extent to general case by "abstract nonsense"

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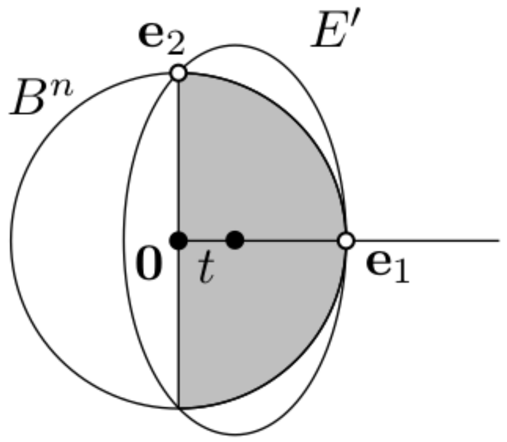

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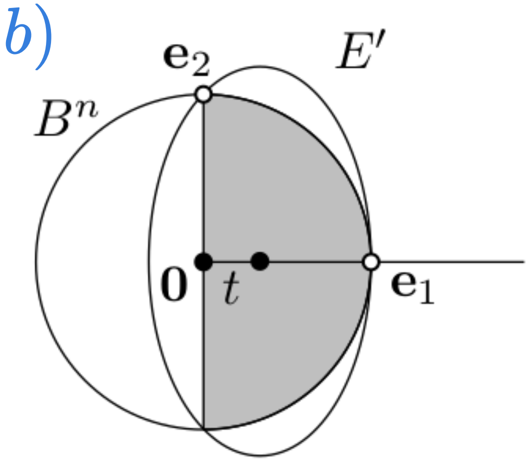

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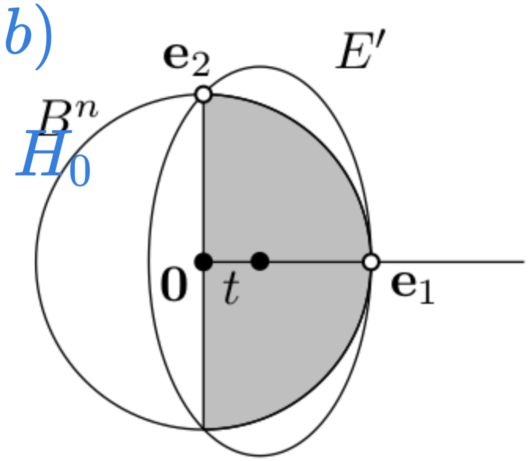

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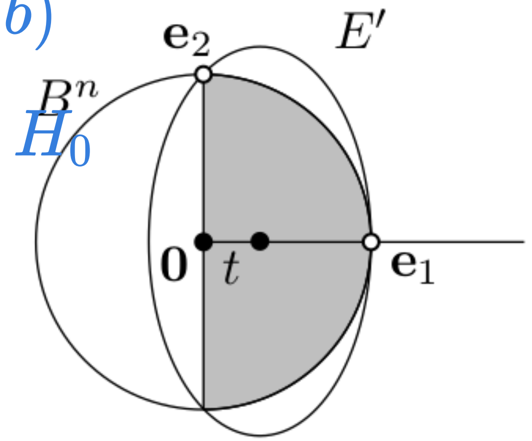
note $E_0' = \{y \mid (y - t \cdot e_1)^\top M^{-2} (y - t \cdot e_1) \leqslant 1\}$

the conditions ensure $e_1, e_2 \in E_0'$ ...



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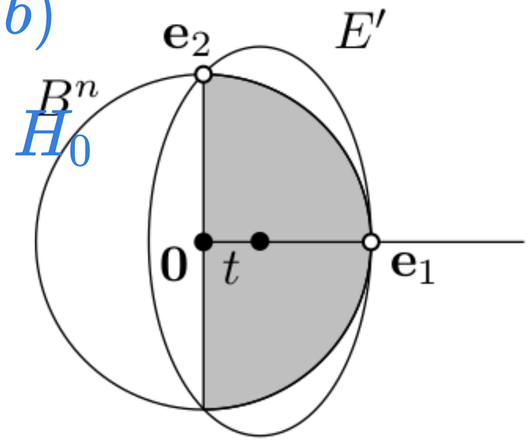
claim: minimum value of $\det M = ab^{n-1}$
subject to these conditions satisfies $\leqslant e^{-1/(2n+2)}$



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feasibility: relaxed vs non-relaxed

last time: poly-time algorithm for *relaxed feasibility problem*

given: $R > \varepsilon > 0$, polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

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it follows that the *relaxed-feasibility algorithm* finds solution to $\{\tilde{A}x \leqslant \tilde{b}\}$ if and only if $\{Ax \leqslant b\}$ is feasible $\rightsquigarrow$ can decide feasibility of $\{Ax \leqslant b\}$ in polynomial time

last time: poly-time algorithm for *relaxed feasibility problem*

given: $R > \varepsilon > 0$, polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$

promise: $\exists y \in \mathbb{R}^n.\ \mathrm{Ball}(y, \varepsilon) \subseteq P \subseteq \mathrm{Ball}(0, R)$

goal: find some point $y \in P$

first step: algorithm for the (non-relaxed) *feasibility problem*

given: system of linear inequalities $\{Ax \leqslant b\}$

goal: decide if system has a solution or not

strategy: compute new system $\{\tilde{A}x \leqslant \tilde{b}\}$ such that

1. if $\{Ax \leqslant b\}$ feasible, then $\{\tilde{A}x \leqslant \tilde{b}\}$ satisfies *relaxed feasibility promise* for R, ε with $\langle R \rangle + \langle \varepsilon \rangle \leqslant \mathrm{poly}(\langle A \rangle, \langle b \rangle)$
2. if $\{Ax \leqslant b\}$ infeasible, then $\{\tilde{A}x \leqslant \tilde{b}\}$ infeasible

remains to show: construction of $\{\tilde{A}x \leqslant \tilde{b}\}$

not-too-large promise

lemma: $\{Ax \leqslant b\}$ feasible if and only if the following system is feasible for $R = 100^{\langle A \rangle + \langle b \rangle}$,

$$\{Ax \leqslant b, -R \cdot \mathbf{1} \leqslant x \leqslant R \cdot \mathbf{1}\}$$

lemma follows from the fact that feasible systems of linear inequalities always have solutions with *small encoding size* *not-too-small promise*

not-too-small promise

let $\{Ax \leqslant b\}$ be a system of linear inequalities

for $\eta \geqslant 0$, let P_η be set of solutions of $\{Ax \leqslant b + \eta \cdot \mathbf{1}\}$

not-too-small promise

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lemma: $\exists \eta, \varepsilon > 0$ with $\langle \eta \rangle + \langle \varepsilon \rangle \leqslant \mathrm{poly}(\langle A \rangle, \langle b \rangle)$ and

1. if $P_0 \neq \emptyset$, then $\mathrm{Ball}(y, \varepsilon) \subseteq P_\eta$ for some $y \in \mathbb{R}^n$
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then, $Ax = Ay + A(x - y) \leqslant b + \|A(x-y)\|_\infty \cdot \mathbf{1}$

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therefore, $Ax \leqslant b + \eta \cdot \mathbf{1}$ whenever $\|x-y\|_2 \leqslant \eta \cdot 2^{-\langle A \rangle}$ $\square$

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proof: $(b + \eta \cdot \mathbf{1})^\intercal y \leqslant -1 + \eta \cdot 2^{O(\langle y \rangle)} < 0$ for η small enough

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so far: we saw a poly-time algorithm for *deciding* the feasibility of systems of linear inequalities

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turns out: if we found such a set, the linear equations $\{Ax = b, x_S = 0\}$ uniquely determine a feasible solution x for the given system

feasible vs optimal solutions

$$\text{maximize } c^\intercal x \text{ subject to } Ax \leqslant b \text{ and } x \geqslant 0 \quad \text{(P)}$$

last step: turn our algorithm for finding a *feasible solution* to a linear program into algorithm for finding an *optimal solution*

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concretely, find feasible solutions x and y for the following system—guaranteed to be optimal solutions for (P) and (D)

$$\{c^\top x = b^\top y,\ Ax \leqslant b, x \geqslant 0,\ A^\top y \geqslant c, y \geqslant 0\}$$