ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Bernd Gärtner, Mohsen Ghaffari, Angelika Steger, David Steurer

Algorithms, Probability, and Computing **Exercises KW42** **HS19**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 13–15 CAB G 56

  Group B: Wed 16–18 NO C 44

  Group C: Wed 16–18 CAB G 52

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise class on October 16, 2019. Please hand in your solutions not later than October 15.

Exercise 1

For a permutation π on the keys $\{1..n\}$, let T_π be the search tree that we obtain from inserting all n keys, one after the other, in the order given by π.

Prove: If π is drawn uniformly at random, then T_π has the same distribution as $\tilde{\mathcal{B}}_{[n]}$ from the lecture.

Exercise 2

Given a sorted sequence $a_0 < a_1 < \cdots < a_{n-1}$ of n real numbers, we consider the convex polygon C with vertices $\left((a_i, a_i^2)\right)_{i=0}^{n-1}$. For $k \in \mathbf{R}$, show that the line with equation $y = 2kx - k^2$ intersects C iff $k \in \{a_0, a_1, \ldots, a_{n-1}\}$.

REMARK: This exercise is supposed to exhibit that deciding whether a line intersects a convex polygon cannot be easier than deciding whether a query key is in a given set of keys.

Exercise 3

We are given a set P of n points in $\mathbf{R}^2$ and a point q which has distinct distances to all points in P. We add the points of P in random order (starting with the empty set), and observe the nearest neighbor of q in the set of points inserted so far. What is the expected number of distinct nearest neighbors that appear during the process?

Exercise 4

Recall the algorithm for counting the points below a query line: We interpreted the question in the dual setting, so we were looking for the number of lines above a query point. Prove the space bound in lemma 3.6, which was left as an exercise:

$$\sum_{\nu \text{ inner node}} |\bar{S}_\nu| \;\leq\; 2n^2$$

where, as you may recall,

- n is the number of lines,
- ν ranges over the nodes of the data structure used in the algorithm, which has one node for every level of the line arrangement;
- S_ν is the set of x-coordinates of the corresponding level;
- $\bar{S}_\nu$ is the 'enhanced' set: If a node ν has no child which is an inner node, $\bar{S}_\nu = S_\nu$. Otherwise, $\bar{S}_\nu$ is obtained from S_ν by adding every other value from each of the sets $\bar{S}_u$, u a non-leaf child of ν.