Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Emo Welzl, Angelika Steger, Peter Widmayer

1/19/2008

Algorithms, Probability, and Computing — *Fall 08 Exam*

Candidate:

First name: ....

Last name: ....

Registration Number: ....
(Stud.-Nr.)

I confirm with my signature that I take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions:

1. You can solve the 6 assignments in any order.
2. Check your exam documents for completeness (1 title sheet and 1 sheet containing 6 assignments).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No additives (Hilfsmittel) allowed.
6. Attempts to defraud yield to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in articulative sentences and keywords. Unfounded or incomprehensible solutions will not be awarded.**

   You can write your solution in English or German.

9. Make sure to write your name on all the sheets.

Good luck!

	achieved points (maximum)	reviewer's signature
1	(12)	
2	(12)	
3	(12)	
4	(12)	
5	(12)	
6	(12)	
Σ	(72)	

Exercise 1

In this exercise we consider the history graph of a trapezoidal decomposition. Let S be a set of n non-crossing segments in general position in the plane and let $\mathcal{T}(S)$ denote the trapezoidal decomposition of S. *Locating* a point q means finding the trapezoid of $\mathcal{T}(S)$ containing q. Suppose that we insert the segments of S one by one, in random order (u.a.r. from all permutations of S).

1. In the lecture it was shown that the expected number of steps for locating a given point q with the history graph structure is at most $4H_n$ (by a *step* we mean following an arrow of the history graph). Reproduce the corresponding proof (you can omit the descriptions of the trapezoidal decomposition and the history graph).
2. For general n describe a particular set S in general position such that the expected number of steps for locating a given point q (with the history graph structure) is at most $2H_n$. Justify your answer!
3. Prove or disprove: For every set S of n non-crossing segments in general position in the plane there is a point $q \in \mathbb{R}^2$ such that the expected number of steps for locating q (with the history graph structure) is at most H_n.

Exercise 2

For this exercise, suppose that $P \neq \mathrm{NP}$.

1. Prove or disprove:
   (a) $PCP(1000, O(1)) = P$
   (b) $PCP(n, O(1)) = P$
   (c) $PCP(\frac{\log \log n}{2}, 10\sqrt{\log n}) = P$ (here log denotes the binary logarithm)
2. In the lecture we have dealt with 3-CNF formulas, such as $(x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_2} \vee x_3 \vee x_4)$. In this exercise we consider so called *3-DNF formulas*, such as $(x_1 \wedge x_2 \wedge \bar{x_3}) \vee (\bar{x_1} \wedge \bar{x_4} \wedge x_5)$. Note that here the logical "and" is used inside the clauses whereas the logical "or" is used outside the clauses. We say that a boolean formula $\mathcal{G}$ is *falsifiable* if there is an assignment α such that α does not satisfy $\mathcal{G}$. Prove the following: There is no polynomial-time-algorithm for the following task
   INPUT: An arbitrary 3-DNF formula $\mathcal{F}$ with m clauses
   OUTPUT: "YES" or "NO"
   - If $\mathcal{F}$ is falsifiable then output "YES"
   - If every assignment satisfies at least $0.05m$ clauses of $\mathcal{F}$ then output "NO"
   (- Otherwise output either "YES" or "NO")

   Hint 1: You can use any Theorem from the script, in particular the approximation version of the PCP Theorem.
   Hint 2: Note: If $\mathcal{G}$ is a 3-CNF then $\neg\mathcal{G}$ is a 3-DNF which is falsifiable if and only if $\mathcal{G}$ is satisfiable.

Exercise 3

1. Let (G, s, t, c) be a network and let f be a maximum flow in this network. Show that if the residual network G_f is not acyclic, then f is not a unique maximum flow.

2. We call a network a *unit network* if the capacity of every edge is 1 (multiple edges are not allowed). Recall that a flow f is *integral* if $f(e)$ is an integer for all edges e. Prove the following statements for every $m, n \geq 3$.

   (a) There is a unit network $N = (G, s, t, c)$ with m edges such that both
   - there is only one minimum s-t-cut
     and
   - there are at least $\frac{m-1}{2}$ integral maximum flows.

   (b) There is a unit network $N = (G, s, t, c)$ with m edges such that both
   - there is only one minimum s-t-cut
     and
   - there are at least $2^{\frac{m-1}{4}}$ integral maximum flows.

   (c) There is a unit network $N = (G, s, t, c)$ with n nodes (including s and t) such that both
   - there is only one integral maximum flow
     and
   - there are at least $n - 1$ minimum s-t-cuts

   (d) There is a unit network $N = (G, s, t, c)$ with n nodes (including s and t) such that both
   - there is only one integral maximum flow
     and
   - there are at least 2^{n-2} minimum s-t-cuts.

Exercise 4

Let v be a node of a binary search tree and let t_{left} and t_{right} denote the size of v's left subtree and the size of v's right subtree, respectively. v is in *equilibrium* if $t_{\text{right}} \in \{t_{\text{left}}, t_{\text{left}} + 1\}$. Note that leaves are always in equilibrium. In the following we consider random search trees on n elements.

1. Determine the probability that the root is in equilibrium. Hint: Distinguish the two cases "n is even" and "n is odd".
2. For odd $n \geq 3$ determine the probability that both the root and its children are in equilibrium.
3. Determine the expected number of nodes which are in equilibrium.

Exercise 5

Let A, A' be two $n \times n$ matrices which differ in at least one entry per row. (I.e., for every $i \leq n$ there is a j such that $A_{i,j} \neq A'_{i,j}$.)

1. Suppose that the entries of A, A' are elements of $GF(2)$ and let B be an $n \times n$ matrix whose entries are chosen u.a.r. from $GF(2)$. What is the expected number of entries where AB and $A'B$ differ? We assume here that everything is computed in $GF(2)$.
2. What is the answer to (i) if we replace $GF(2)$ with $GF(5)$?
3. Let x be a 5-dimensional vector whose entries are chosen u.a.r. from $GF(2)$. Give two 5×5 matrices A, A' differing in at least one entry per row such that $\Pr[Ax = A'x] = \frac{1}{32}$.

Exercise 6

In the lecture we considered Karger's basic min-cut algorithm $\mathcal{A}$.

1. In the lecture it was shown that $\Pr[\mathcal{A} \text{ is successful}] \geq \frac{2}{n(n-1)}$. Give a proof for this.
2. Prove that for $n \geq 3$ the inequality in (i) is strict, i.e., $\Pr[\mathcal{A} \text{ is successful}] > \frac{2}{n(n-1)}$.
3. Prove or disprove: For every n there is a connected graph G of n vertices such that $\mathcal{A}$ finds a min-cut of G with probability 1.
   Hint: Remember the upper bound on the number of min-cuts shown in one of the exercise sets.