ETH
Eidgenössische Technische Hochschule Zürich
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Institute of Theoretical Computer Science
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Algorithms, Probability, and Computing — Midterm Exam — HS17

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (9 one-sided pages with 6 exercises).
2. You have 2 hours to solve the exercises.
3. You can solve the exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
4. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
5. Pencils are not allowed. Pencil-written solutions will not be reviewed.
6. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
7. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
8. Provide only one solution to each exercise. Cancel invalid solutions clearly.
9. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
10. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
11. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(10)	
2	(10)	
3	(10)	
4	(10)	
5	(5)	
6	(10)	
Σ	(55)	

Exercise 1 - Five short questions (10 points)

There are 5 subparts, each worth 2 points. Give short answers. There are no negative points for wrong answers. Only questions (c)-(e) have true/false claims.

(a) Given a point set with n points, what are the storage space and query time for the data structure that uses point-line duality and solves the problem of counting the number of points below a query line?

Storage space: ....

Query time: ....

(b) Let T be a binary search tree with keys $[n]$. Define for every node v in T the *weight* $w(v)$ as the number of nodes in the subtree of v (v inclusive). In terms of the weights, what is the probability that a random binary search tree on keys $[n]$ is equal to T?

Answer: ....

....

(c) Let S be a set of $n \geq 2$ segments in the plane that are non-crossing, i.e., whose relative interiors are disjoint. Then the set of endpoints $P(S)$ has size at least $|P(S)| \geq n - 1$.

[ ] False [ ] True

Justification: ....

....

(d) If a graph G has two edge-disjoint minimum cuts, then the random contraction algorithm BasicMinCut(G) returns the size of the minimum cut in G with probability 1.

[ ] False [ ] True

Justification: ....

....

(e) Let p be the polynomial

$$p = \det \begin{pmatrix} 0 & x_{12} & 0 & 0 \\ x_{21} & 0 & x_{23} & 0 \\ 0 & x_{32} & 0 & x_{34} \\ 0 & x_{42} & x_{43} & 0 \end{pmatrix}.$$

Then p is the zero polynomial.

[ ] False [ ] True

Justification: ....

....

Exercise 2 - Parental relationships (10 points)

Let $i \geq 2$ and consider a random search tree on keys $[n]$ where $n \geq i$. Let $P_1^i = P_1^i(n)$ be the indicator random variable for the event that node i is a parent of node 1. (Note: In the special assignment we considered $P_i^1(n)$ instead of $P_1^i(n)$.)

Compute the probability $\Pr[P_1^i(n) = 1]$.

Exercise 3 - Treaps (10 points)

(a) Consider the permutation $\pi = (5, 3, 2, 1, 4)$ of five elements and the corresponding sequence of priorities $p = (0.30, 0.17, 0.50, 0.25, 0.20)$. Draw the sequence of 5 treaps that result when inserting elements into an empty treap in the order given by π with respective priorities given by p. You don't need to further explain your drawings.

(b) Let T be a randomized search tree in which the key i is the last element that was inserted. How to read off from T how many rotations were done when inserting key i? You don't need to justify your answer.

Exercise 4 - Orthonormal vectors (10 points)

Your friend gives you k vectors $v_1, \ldots, v_k \in \mathbb{Q}^n$ and claims that they are orthonormal, i.e., that $v_i^T v_j = 0$ if $i \neq j$ and $v_i^T v_j = 1$ if $i = j$. You are suspicious of his claim and want to verify it.

(a) Describe an efficient algorithm that detects with probability at least $\frac{1}{2}$ whether your friend is lying. Remember to analyze its runtime. More efficient algorithms give more points.

(b) Let $\delta \in (0, 1)$ and let $\mathcal{A}$ be the algorithm from (a). How to use $\mathcal{A}$ to devise an algorithm $\mathcal{A}'$ that detects a lie with probability at least $1 - \delta$?

Exercise 5 - Minimum Cut (5 points)

Consider the multigraph $G = (\{v_1, v_2, v_3, v_4\}, \{a, b, c, d, e, f\})$ in Figure 1. What is the probability that the random contraction algorithm BASICMINCUT returns the size of the minimum cut?

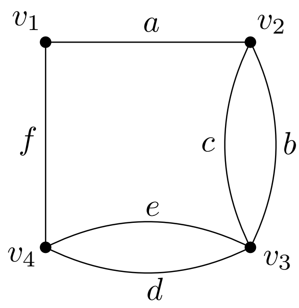

Figure 1: The graph G for Exercise 5.

Exercise 6 - Arrangement of circles (10 points)

Let $C = \{C_1, \ldots, C_n\}$ be an arrangement of n circles in the plane with the property that every circle intersects with at most 5 of the other circles. The *cells* of the arrangement are the maximal open connected regions that remain after removing the boundaries of the circles in C. Let $\mathcal{T}(C)$ be the set of all cells in the arrangement C. See Figure 2 below for an example.

(a) Show that the number of cells in $\mathcal{T}(C)$ is $O(n)$.

Let π be a uniformly random permutation of $[n]$ and insert the cycles to an initially empty arrangement in the order given by π, i.e., in the order $C_{\pi(1)}, \ldots, C_{\pi(i)}$. Let $\mathcal{S}$ be the set of all cells that appear in at least one of the intermediate arrangements $\{C_{\pi(1)}, \ldots, C_{\pi(i)}\}$ where $i = 0, \ldots, n$. Assume that for any choice of π the boundaries of all cells in $\mathcal{S}$ are described by at most r many circular arcs where r is a constant.

(b) Let $q \in \mathbb{R}^2$ be fixed, and assume that q is not on the boundary of any circle in C. Let N denote the number of cells in $\mathcal{S}$ that contain q. Show that $\mathbb{E}[N] \in O(\log n)$.

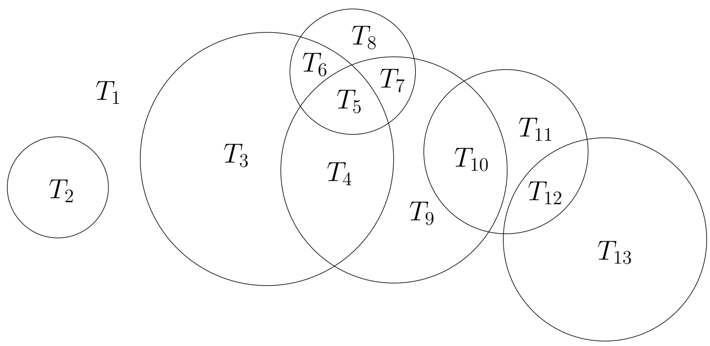

Figure 2: An example arrangement of circles. The cells of this arrangement are marked with $T_1, \ldots, T_{13}$.