

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

Group A: Wed 14–16

Group B: Wed 14–16

Group C: Wed 16–18

Group D: Wed 16–18

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer”, then a formal proof is **always** required.
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The following exercises will be discussed in the exercise classes on November 14, 2021. You can hand in your solutions during the lecture break on November 12 or November 13 or via Moodle, no later than 4 pm at November 13.

Exercise 1

Show that every feasible point of the Tight Spanning Tree LP is feasible in the Loose Spanning Tree LP – without using theorem 4.11.

Exercise 2

Consider the following linear program, almost the Tight Spanning Tree LP, it seems:

Some LP for graph $G = (V, E)$, $c \in \mathbb{R}^E$

$\min c^T x$

subject to

$$\begin{aligned} \sum_{e \in E} x_e &= n \\ \sum_{e \in E \cap \binom{S}{2}} x_e &\leq |S| - 1, \text{ for all } S \subseteq V, \emptyset \neq S \neq V, \text{ and} \\ 1 \geq x_e &\geq 0, \text{ for all } e \in E. \end{aligned}$$

What are the edge sets corresponding to vectors $x \in \{0, 1\}^E$ feasible in Some LP?

Exercise 3

Let $D = (V, A)$ be a directed graph and let $s, t \in V$. To any vertex set $S \subseteq V$ we associate a *cut* $C(S) \subseteq A$ that consists of all arcs between S and $V \setminus S$. We say that $C(S)$ is an *s-t cut* if $s \in S$ and $t \notin S$. We say that $C(S)$ is a *strong s-t cut* if it is an s-t cut and if all edges in $C(S)$ are directed away from $V \setminus S$. See Figure 1 for an example.

In this exercise we will prove the following lemma and see that it is a special case of the Farkas lemma we have seen in the lecture. Informally, it says that there is a simple certificate for both proving and disproving the existence of a directed s-t path in D .

Lemma 1 (Farkas lemma for s-t-paths). *Exactly one of the following two statements holds for any directed graph $D = (V, A)$ and for any two vertices $s, t \in V$.*

- i) *There exists a directed s-t path.*
- ii) *There exists a strong s-t cut.*

For every vertex $v \in V$ let $\delta(v)^+ \subseteq A$ denote the arcs that are outgoing from v and let $\delta(v)^- \subseteq A$ denote the arcs that are incoming to v .

- (a) Show that there is a directed s-t path in D if and only if the following system of equations and inequalities has a solution over the real valued variables $\{x_e \mid e \in A\}$.

$$\forall v \in V: \sum_{e \in \delta(v)^+} x_e - \sum_{e \in \delta(v)^-} x_e = \begin{cases} 0 & \text{if } v \in V \setminus \{s, t\} \\ 1 & \text{if } v = s \\ -1 & \text{if } v = t \end{cases}$$

$$\forall e \in A: x_e \geq 0$$

- (b) Prove Lemma 1 by applying some version of Farkas lemma to the system in (a).
- (c) Prove Lemma 1 directly without using (a) or Farkas lemma.

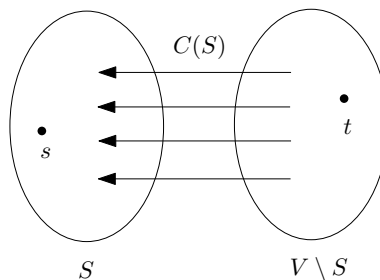


Figure 1: An illustrative example of a strong s-t cut. The cut $C(S)$ is a strong s-t cut because all edges in $C(S)$ are directed away from $V \setminus S$.

Exercise 4

Suppose we are running the checking algorithm for matrices over $\text{GF}(2)$, i.e. numbers are $\{0, 1\}$ with addition and multiplication $\pmod 2$. Show that in one iteration the success probability of detecting an error in the supposed product matrix C is exactly $\frac{1}{2}$, in case matrix C is wrong in exactly one row.

Exercise 5

For $n \in \mathbf{N}$, let $A \in \mathbf{R}^{n \times n}$ be a non-zero matrix (i.e. not all entries are 0) and let x be a vector u.a.r. from $\{-1, 0, +1\}^n$. Show that the probability that the vector Ax is non-zero is at least $2/3$.

Exercise 6

Given a finite set S of rational numbers and positive integers d and n , $d \leq |S|$, find a polynomial $p(x_1, x_2, \dots, x_n)$ of degree d for which the Schwartz–Zippel theorem is tight. That is, the number of n -tuples $(r_1, \dots, r_n) \in S^n$ with $p(r_1, \dots, r_n) = 0$ is $d|S|^{n-1}$.