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Algorithms, Probability, and Computing **Solutions for SPA 2** **HS21**

Solution 1

(a) Consider some edge $e \in E$. Note that if e is tight with respect to x_i, then it is also tight with respect to $x_{i+1}, x_{i+2}, \ldots$. Hence, it suffices to show that e gets tight eventually. This is clearly the case as otherwise $x_e^{i+1} = 2x_e^i$ for $i \geq 0$, which implies that there exists some i with $x_e^i \geq 0.5$. This in turn would imply that e is tight with respect to x^i, a contradiction.

(b) We show by induction on i that x^i is a feasible solution of the linear program. We have $x^0 \geq \mathbf{0}$ by definition. Now, consider an arbitrary $v \in V$. It holds that

$$\sum_{e \in E\colon v \in e} x_e^0 = |e \in E\colon v \in e| \cdot \frac{1}{n} \leq 1,$$

as desired. Now, consider an arbitrary i and assume that x^i is feasible. We show that this implies that x^{i+1} is feasible. Non-negativity of x^{i+1} directly follows from the non-negativity of x^i. Now, consider an arbitrary vertex $v \in V$. If v is tight, then

$$\sum_{e \in E\colon v \in e} x_e^{i+1} = \sum_{e \in E\colon v \in e} x_e^i \leq 1,$$

as desired. On the other hand, if v is not tight then

$$\sum_{e \in E\colon v \in e} x_e^{i+1} \leq \sum_{e \in E\colon v \in e} 2x_e^i \leq 2 \cdot \frac{1}{2} = 1$$

and therefore x^{i+1} is a feasible solution.

(c) The dual of the linear program is

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V} y_v & \\ \text{subject to} & y_u + y_v \geq 1 & \text{for every } \{u, v\} \in E \\ & y \geq \mathbf{0}. & \end{array} \tag{1}$$

(d) For every $v \in V$, we set $y'_v = 1$ if v is tight with respect to x' and $y'_v = 0$ otherwise. We first show that y' is a dual feasible solution. By definition, y' is non-negative. Now, let $e = \{u, v\} \in E$ be an arbitrary edge. Note that e is tight with respect to x' and hence at least one of its endpoints is tight, which directly implies $y_u + y_v \geq 1$. It remains to verify that $\sum_{v \in V} y'_v \leq 4 \sum_{e \in E} x'_e$. We have

$$\sum_{v \in V} y'_v = \sum_{v \text{ is tight with respect to } x'} 1 \leq \sum_{v \text{ is tight with respect to } x'} 2 \sum_{e \in E:\, v \in e} x'_e \leq 2 \sum_{v \in V} \sum_{e \in E:\, v \in e} x'_e.$$

Note that for each $e \in E$, x'_e is counted twice in the last expression, once for each endpoint, and therefore $2 \sum_{v \in V} \sum_{e \in E:\, v \in e} x'_e \leq 4 \sum_{e \in E} x'_e$. Hence, $\sum_{v \in V} y'_v \leq 4 \sum_{e \in E} x'_e$, as desired.

(e) Weak duality implies that $\sum_{v \in V} y'_v \geq \mathrm{OPT}$. Hence, together with the previous exercise we obtain $\mathrm{OPT} \leq 4 \sum_{e \in E} x'_e$ and therefore $\sum_{e \in E} x'_e \geq \frac{1}{4}\mathrm{OPT}$.

(f) Consider an arbitrary $\varepsilon \in (0, 1]$. From now on, we say that a vertex is tight with respect to x if We now say that a vertex is tight with respect to x if $\sum_{e \in E:\, v \in E} x_e \geq 1 - \frac{\varepsilon}{4}$. Note that we still say that an edge is tight if one of its endpoints is tight. We now consider the following sequence of vectors $x^0, x^1, \ldots \in \mathbb{R}^E$: for each $e \in E$, $x^0_e = \frac{1}{|V|}$ and for $i \geq 0$ we set

$$x^{i+1}_e = \begin{cases} x^i_e & \text{if } e \text{ is tight with respect to } x^i \\ \left(1 + \frac{\varepsilon}{4}\right) x^i_e & \text{otherwise.} \end{cases}$$

The proof that there exists a j with $x^j = x^{j+1} = x^{j+2} = \ldots$ is completely analogous to the proof of a). Similarly, the proof that x' is a primal feasible solution is completely analogous to the proof of b), using the fact that $\left(1 + \frac{\varepsilon}{4}\right)\left(1 - \frac{\varepsilon}{4}\right) \leq 1$. It remains to show that $\sum_{e \in E} x'_e \geq \frac{1}{2+\varepsilon}\mathrm{OPT}$. To that end, consider y' with $y'_v = 1$ if v is tight with respect to x' and $y'_v = 0$ otherwise. Similar as before, y' is a dual feasible solution. Hence,

$$\begin{aligned}
\sum_{e \in E} x'_e &= \frac{1}{2} \sum_{v \in V} \sum_{e \in E:\, v \in e} x'_e \\
&\geq \sum_{v \text{ is tight with respect to } x'} \frac{1}{2} \sum_{e \in E:\, v \in e} x'_e \\
&\geq \sum_{v \text{ is tight with respect to } x'} \frac{1}{2}\left(1 - \frac{\varepsilon}{4}\right) \\
&\geq \frac{1}{2}\left(1 - \frac{\varepsilon}{4}\right) \sum_{v \in V} y'_v \\
&\geq \frac{1}{2+\varepsilon}\mathrm{OPT},
\end{aligned}$$

where the last inequality follows from weak duality together with the fact that $\frac{1}{2}\left(1 - \frac{\varepsilon}{4}\right) \geq \frac{1}{2+\varepsilon}$.

Solution 2

(a) Consider the following Integer Linear Program

$$\begin{array}{lcc} \text{maximize} & \sum_{v \in V} y_v & \\ \text{subject to} & y_u + y_v \leq 1 & \forall \{u, v\} \in E \\ & y_v \in \{0, 1\} & \forall v \in V. \end{array} \tag{2}$$

Let $y \in \{0,1\}^V$ denote an arbitrary feasible solution to the ILP. Now, let $S := \{v \in V : y_v = 1\}$. To prove that S is a valid independent set, consider an arbitrary edge $\{u, v\} \in E$. As $y_u + y_v \leq 1$, we have $y_v = 0$ or $y_u = 0$. Hence, u and v cannot simultaneously be in S. Moreover, we have $|S| = \sum_{v \in V} y_v$. Now, let S denote an arbitrary independent set. Let $y \in \{0,1\}^V$ with $y_v = 1$ if $v \in S$ and $y_v = 0$ otherwise. For every edge $\{u, v\} \in E$, u is not in the independent set or v is not in the independent set and therefore $y_u + y_v \leq 1$. Hence, y is a feasible solution with $\sum_{v \in V} y_v = |S|$.

(b) We relax the ILP as follows

$$\begin{array}{lcc} \text{maximize} & \sum_{v \in V} y_v & \\ \text{subject to} & y_u + y_v \leq 1 & \forall \{u, v\} \in E \\ & 0 \leq y_v \leq 1 & \forall v \in V. \end{array} \tag{3}$$

The size of the maximum independent set of the complete graph on n nodes is 1. Now, consider y with $y_v = \frac{1}{2}$ for every $v \in V$. Then, y is a valid solution to the relaxed linear program with $\sum_{v \in V} y_v = \frac{n}{2}$. Hence, the integrality gap is $\frac{n}{2} = \Omega(n)$.

(c) It clearly holds that $|S \cap C| \leq \frac{|C|-1}{2}$, as otherwise strictly more than half of the vertices in the cycle would be contained in S which in turn would imply that there exists two neighboring nodes that are contained in S, a contradiction.

We consider the following strengthened linear program.

$$\begin{array}{lcc} \text{maximize} & \sum_{v \in V} y_v & \\ \text{subject to} & y_u + y_v \leq 1 & \forall \{u, v\} \in E \\ & \sum_{v \in C} y_v \leq \frac{|C|-1}{2} & \text{for every odd cycle } C \\ & 0 \leq y_v \leq 1 & \forall v \in V. \end{array} \tag{4}$$

One can again consider the complete graph on n nodes, this time setting $y_v = \frac{1}{3}$ for every $v \in V$. The vector y is a valid solution for the tightened LP and therefore the integrality gap is $\frac{n}{3} \geq C$ for a sufficiently large n.

(d) Let $G' = (V', E')$ be the graph whose vertex set contains two copies of each vertex in V and for each edge $\{u, v\} \in E$ there are two edges in E': one between the first copy of u and the second copy of v and one between the second copy of u and the first copy of v. For a given vertex $v \in V$, we denote with P_v the shortest path that connects the two copies of v in G' with each other, assuming that the two copies are connected in G'. Now, let v^* be the vertex for which P_{v^*} has the shortest length (among vertices whose two copies are connected). Note that P_{v^*} corresponds to an odd cycle C^* in G with the weight of the cycle being equal to the length of the path P_{v^*}. We claim that $w(C^*) = \min_{C \text{ is an odd cycle}} w(C)$. For the sake of contradiction, assume there exists an odd cycle C' with $w(C') < w(C^*)$. It is easy to see that for every $v' \in C'$, the length of the path $P_{v'}$ is not larger than $w(C')$, but together with $w(C') < w(C^*)$ this contradicts the assumption that v^* is the vertex for which P_{v^*} has the shortest length. It therefore only remains to show that one can compute P_v for every vertex $v \in V$ in polynomial time. This can be done by using Dijkstra's shortest path algorithm n times, using the fact that all the edges have non-negative weight.

(e) Consider an arbitrary $y \in \mathbb{R}^V$. It is easy to verify in polynomial time if y is non-negative, to check if $y_u + y_v \leq 1$ for every $\{u, v\} \in E$ and to return a violated constraint if this is not the case. It remains to handle the odd cycle constraints. To that end, we assign each edge $e = \{u, v\} \in E$ the weight $w_e = 1 - y_u - y_v$. Note that $w_e \geq 0$ if all of the other constraints are satisfied. Hence, we can use the algorithm from above to find a minimum weight odd cycle in G with respect to w, which we denote by C^*. Assume that $w(C^*) < 1$. By the definition of w, this implies $|C^*| - 2\sum_{v \in C^*} y_v < 1$ and by rearranging terms this implies $\sum_{V \in C^*} y_v > \frac{|C^*|-1}{2}$. Hence, we have found a violated constraint. On the other hand, if $w(C^*) \geq 1$, then all of the cycle constraints are satisfied by y. This follows as for each odd cycle C whose constraint is violated we have $w(C) < 1$.

Solution 3

We first show $\mathrm{rk}(A) \geq 2k$. To that end, let M be an arbitrary matching in G of size k. Let $I = \{i \in \{1, 2, \ldots, n\} \colon M \text{ matches } v_i\}$ and $V' = \{v \in V \colon M \text{ matches } v\}$. Note that $A^{I,I}$ is the Tutte matrix of $G[V']$. Moreover, $G[V']$ is a graph on $2k$ vertices which contains a perfect matching. We showed in the lecture that this implies that $\det(A^{I,I})$ is not the zero-polynomial and therefore $\mathrm{rk}(A^{I,I}) = 2k$, which directly implies that $\mathrm{rk}(A) \geq 2k$.
It remains to show $\mathrm{rk}(A) \leq 2k$. There exists index sets I and J with $|I| = |J| = \mathrm{rk}(A)$ such that $A^{I,J}$ and $A^{J,I} = -\left(A^{I,J}\right)^T$ have full-rank. Hence, the hint implies that $\det(A^{I,I}) \cdot \det(A^{J,J}) = \det(A^{I,J}) \cdot \det(A^{J,I}) \neq 0$. Therefore, $\det(A^{I,I}) \neq 0$. Let $V' = \{v_i \colon i \in I\}$. Then, $A^{I,I}$ is the Tutte matrix of $G[V']$. AS $A^{I,I}$ has full-rank, we have seen in the lecture that $G[V']$ has a perfect matching. This matching has a size of $\mathrm{rk}(A)/2$ and its size gives a lower bound on the size of the largest matching in G. Therefore, $\mathrm{rk}(A) \leq 2k$ which together with $\mathrm{rk}(A) \geq 2k$ implies $\mathrm{rk}(A) = 2k$, as desired.