ETH zürich

Institute of Theoretical Computer Science
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Algorithms, Probability, and Computing **Special Assignment 2** **HS21**

- Write an exposition of your solution using a computer, where we strongly recommend to use LaTeX. **We do not grade hand-written solutions**.
- You need to submit your solution via Moodle until **Tuesday, December 7, 2021** by **10 pm**. Late solutions will not be graded.
- For geometric drawings that can easily be integrated into LaTeX documents, we recommend the drawing editor IPE, retrievable at `http://ipe.otfried.org` in source code and as an executable for Windows.
- Write short, simple, and precise sentences.
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer" or "justify intuitively", then a formal proof is **always** required. You can of course refer in your solutions to the lecture notes and to the exercises, if a result you need has already been proved there.
- We would like to stress that the ETH Disciplinary Code applies to this special assignment as it constitutes part of your final grade. The only exception we make to the Code is that we encourage you to verbally discuss the tasks with your colleagues. However, you need to write down the names of all your collaborators at the beginning of the writeup. It is strictly prohibited to share any (hand)written or electronic (partial) solutions with any of your colleagues. We are obligated to inform the Rector of any violations of the Code.
- There will be two special assignments this semester. Both of them will be graded and the average grade will contribute 20% to your final grade.
- As with all exercises, the material of the special assignments is relevant for the (midterm and final) exams.

Exercise 1 **35 points**

(*Fractional Matching*)

Let $G = (V, E)$ be a graph. Consider the following linear program with one variable x_e for each edge $e \in E$:

$$\text{maximize} \sum_{e\in E} x_e \quad \text{subject to} \quad \sum_{e\in E: v\in e} x_e \leq 1 \text{ for every } v \in V \text{ and } \mathbf{x} \geq \mathbf{0}.$$

We say that a vertex v is *tight* with respect to $x \in \mathbb{R}^E$ if $\sum_{e\in E: v\in e} x_e \geq 0.5$. Moreover, we say that an edge e is *tight* with respect to x if one or both endpoints of e are tight with respect to x. Consider the following sequence of vectors $x^0, x^1, \ldots \in \mathbb{R}^E$: for each $e \in E$, $x_e^0 = \frac{1}{|V|}$ and for $i \geq 0$ we set

$$x_e^{i+1} = \begin{cases} x_e^i & \text{if } e \text{ is tight with respect to } x^i \\ 2x_e^i & \text{otherwise.} \end{cases}$$

(a) Show that there exists some j such that $x' := x^j = x^{j+1} = x^{j+2} = x^{j+3} = \ldots$.

(b) Show that x' is a feasible solution of the linear program above.

(c) Write down the dual of the linear program above.

(d) Find an explicit dual feasible solution y' with $\sum_{v\in V} y'_v \leq 4 \sum_{e\in E} x'_e$.

(e) Let OPT denote the optimal value of the primal. Show that $\sum_{e\in E} x'_e \geq \frac{1}{4}\text{OPT}$.

(f) Let $\varepsilon \in (0, 1]$ be arbitrary. Perform a *slight* modification to how the sequence of vectors $x^0, x^1, \ldots$ is computed such that there again exists a j such that $x' := x^j = x^{j+1} = x^{j+2} = x^{j+3} = \ldots$ is a primal feasible solution with $\sum_{e\in E} x'_e \geq \frac{1}{2+\varepsilon}\text{OPT}$.

Exercise 2 **40 points**

(*Maximum Independent Set*) Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is an independent set if there does not exist an edge $\{u, v\} \in E$ with $u, v \in S$. The maximum independent set problem asks to find an independent set of maximum size. In the following exercise, we consider linear programming relaxations of the maximum independent set problem.

(a) Devise an integer linear program with the following property: each feasible solution to the integer linear program with objective value z corresponds to an independent set with z vertices and vice versa.

(b) Relax the integer linear program devised in a). Moreover, for each n, find an n-node graph such that the integrality gap of the relaxed linear program is $\Omega(n)$ (i.e., the optimum value of the relaxed linear program divided by the optimum value of the integer linear program is $\Omega(n)$).

(c) Let S be an independent set and C be an odd cycle. Show that $|S \cap C| \leq \frac{|C|-1}{2}$. Use this observation to strengthen your linear program from b) by adding one constraint for each odd cycle. Show that for every fixed constant $c > 1$ there exists a graph for which the resulting linear program has an integrality gap of at least c.
Hint: You can use the fact that for every $k, \ell > 0$ there exists an $n_{k,\ell}$-node graph with a maximum independent set size of at most $\frac{n_{k,\ell}}{k}$ and a girth larger than ℓ (The girth of a graph is defined as the length of the shortest cycle).

(d) The minimum odd cycle problem is defined as follows: the input is a graph with at least one odd cycle together with a weight function w that assigns each edge of the input graph a *non-negative* weight. For a cycle C, we denote with $w(C)$ the total weight of all edges in the cycle. The output is an odd cycle C^* with $w(C^*) = \min_{C \text{ is an odd cycle}} w(C)$.
Devise a polynomial-time algorithm for the minimum odd cycle problem.
Hint: You might want to construct a graph with twice as many vertices as the input graph.

(e) Note that the linear program from exercise c) can have exponentially many constraints. However, we have learned during class that we can solve a linear program with the ellipsoid method in polynomial time as long as there exists a separation oracle that runs in polynomial time. Show that there exists a separation oracle for the linear program devised in c) that runs in polynomial time.

Exercise 3 **25 points**

(*Rank of the Tutte matrix*)

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. We have defined the Tutte matrix A of G by setting

$$a_{ij} = \begin{cases} x_{ij} & \text{if } i < j \text{ and } \{v_i, v_j\} \in E, \\ -x_{ji} & \text{if } i > j \text{ and } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j \in \{1, 2, \ldots, n\}$.

Recall that the rank $\mathrm{rk}(M)$ of a matrix M is the maximum number of linearly independent rows/columns of M. Note that A is not a standard matrix in the sense that some entries are variables. Let S_A denote the set consisting of all matrices that one can obtain from A by fixing the variables in A in an arbitrary way. We define the rank of A as $\mathrm{rk}(A) = \max_{M \in S_A} \mathrm{rk}(M)$. Let k denote the size of the largest matching in G. Show that $\mathrm{rk}(A) = 2k$.
Hint: Let $I, J \subseteq \{1, 2, \ldots, n\}$ be two index sets and M an *antisymmetric* $n \times n$-matrix with $\mathrm{rk}(M) = r$. We denote with $M^{I,J}$ the $|I| \times |J|$ submatrix of M which contains the i-th row if $i \in I$ and the j-th column if $j \in J$. Then, if $|I| = |J| = r$ it holds that $\det(M^{I,I}) \cdot \det(M^{J,J}) = \det(M^{I,J}) \cdot \det(M^{J,I})$.