Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, David Steurer, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing **Midterm Exam** **HS18**

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (8 one-sided pages with 5 exercises).
2. You have **2 hours** to solve the exercises.
3. You can solve the exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
4. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
5. Pencils are not allowed. Pencil-written solutions will not be reviewed.
6. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
7. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
8. Provide only one solution to each exercise. Cancel invalid solutions clearly.
9. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions.**
10. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
11. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

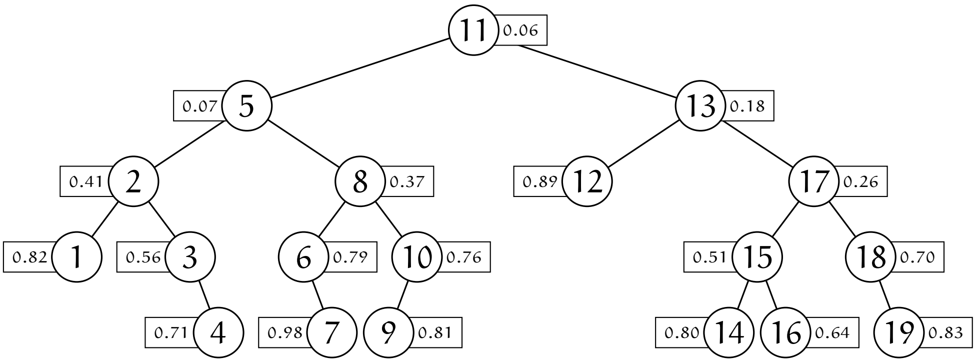
Good luck!

	achieved points (maximum)	reviewer's signature
1	(15)	
2	(8)	
3	(12)	
4	(10)	
5	(15)	
Σ	(60)	

Exercise 1: Five Short Questions (15 points)

No justification is required. Only give final exact answers. (There are no negative points for wrong answers.)

(a) **(3 points)** The node 13 has been inserted into a treap, and the resulting treap is shown below. How many rotations were performed during the insertion of node 13?



Answer: ....

....

(b) **(3 points)** Consider the process of inserting the keys $\{1, 2, \ldots, n\}$ into an empty treap in the order given by the sequence $\langle 1, 2, \ldots, n \rangle$. What is the probability that the rank of the root changes in every step in this process?

Answer: ....

....

(c) **(3 points)** Let G be a weighted graph with m edges. In the course, you have seen a randomized algorithm that computes a minimum spanning tree in time $\mathcal{O}(m)$ in expectation, which we called Randomized Minimum Spanning Tree Algorithm. What is the probability that this algorithm returns a minimum spanning tree for G?

Answer: ....

....

(d) **(3 points)** Assume that S is a set of 9 distinct segments in the plane that are non-crossing, i.e., whose relative interiors are disjoint. Let $P(S)$ be the set of endpoints of the segments in S. What is the minimum and maximum possible size of $P(S)$?

Answer: ....

....

(e) **(3 points)** Consider a polyhedron of the form $C := \{x \in \mathbb{R}^n | Ax \leq 0\}$, $A \in \mathbb{R}^{m \times n}$. What are the possible basic feasible solution(s)?

Answer: ....

....

Exercise 2: Minimum Cut (8 points)

Let G be a multigraph. Assume that in the algorithm BasicMinCut(G), in each round instead of contracting a random edge, we choose two nodes uniformly at random and merge them (see Figure 1) until the remaining graph has two nodes. Prove that for any positive integer n, there exists an n-node multigraph for which the algorithm returns a minimum cut with an exponentially small probability, i.e., $\mathcal{O}(\frac{1}{c^n})$ for some constant $c > 1$.

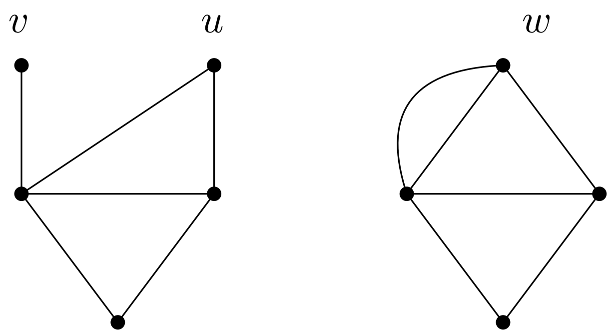


Figure 1: Merging nodes v and u into w. (Notice that merging is the same as contraction if the two nodes share an edge.)

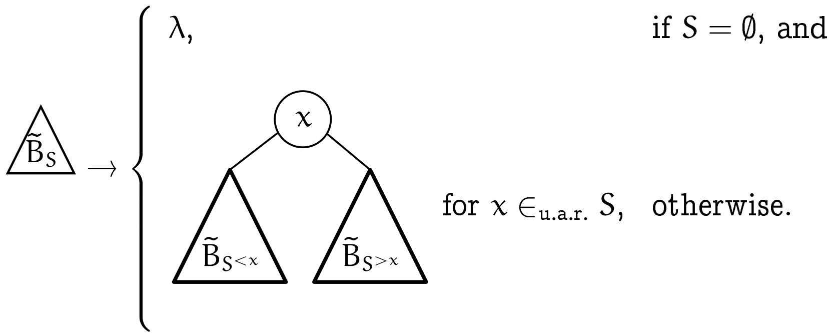
Exercise 3: Random Search Trees (12 points)

(a) **(7 points)** Assume that for $n \geq 2$,

$$a_n = \frac{2}{n(n+1)} \sum_{j=2}^{n} (j a_{j-1} + j)$$

and $a_1 = 0$. Prove that $a_n = 2H_{n+1} - 3$ for $n \geq 3$.

(b) **(5 points)** Recall that for a given set S of $n \in \mathbb{N}_0$ distinct real numbers, we defined random search trees recursively as follows



Assume that instead of choosing key x uniformly at random (i.e., $x \in_{\text{u.a.r.}} S$), we choose $x \in S$ with probability $x / (\sum_{y \in S} y)$.

For set $S = \{1, \cdots, n\}$, find a recurrence formula for the expected depth of the smallest key. (You do not need to solve the recurrence.)

Exercise 4: Point Location (10 points)

We are given a set P of n points in $\mathbb{R}^3$. We define the *perimeter* $d(p_1, p_2, p_3)$ between three points p_1, p_2, p_3 to be the sum of the distances between each pair of them, i.e.,

$$d(p_1, p_2, p_3) := ||p_1 - p_2|| + ||p_2 - p_3|| + ||p_1 - p_3||.$$

Furthermore, assume that no two triples from P have the same perimeter.

We add the points of P in random order to a set Q (which is initially empty). For $i = 3, \ldots, n$ let $\{p_i^{(1)}, p_i^{(2)}, p_i^{(3)}\}$ denote the point triple in Q of minimal perimeter, after the i-th insertion of a point, i.e.,

$$d(p_i^{(1)}, p_i^{(2)}, p_i^{(3)}) = \min_{\{p_1,p_2,p_3\} \in \binom{Q}{3}} d(p_1, p_2, p_3).$$

(For $|Q| < 3$ the triple of minimal perimeter is not defined.)

Compute the expected number of distinct point triples with minimal perimeter that appear during the process. In other words, what is $\mathbf{E}[|\{\{p_i^{(1)}, p_i^{(2)}, p_i^{(3)}\} : i = 3, \ldots, n\}|]$?

Exercise 5: Linear Programming (15 points)

(a) **(5 points)** Assume that $\mathbf{c} \in \mathbb{R}^n$ is some fixed vector. We denote by $\mathbf{1} \in \mathbb{R}^n$ and by $\mathbf{0} \in \mathbb{R}^n$ the vectors of all-ones and all-zeros, respectively. Let x be a vector of n real valued variables. Describe explicitly the set of all basic feasible solutions and find an optimal basic feasible solution for the following linear program.

$$\min \mathbf{c}^\mathsf{T}\mathbf{x} \text{ subject to } \mathbf{1}^\mathsf{T}\mathbf{x} = 2 \text{ and } \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}.$$

(b) **(5 points)** Prove that Proposition 2 holds by applying Proposition 1.

Proposition 1. *A system* $A\mathbf{x} \leq \mathbf{b}$ *of linear inequalities has no solution if and only if there exists a vector* $\mathbf{y} \geq \mathbf{0}$ *such that* $A^\mathsf{T}\mathbf{y} = \mathbf{0}$ *and* $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$.

Proposition 2. *A system* $B\mathbf{x} \leq \mathbf{c}$ *of linear inequalities has no* nonnegative *solution if and only if there exists a vector* $\mathbf{z} \geq \mathbf{0}$ *such that* $B^\mathsf{T}\mathbf{z} \geq \mathbf{0}$ *and* $\mathbf{c}^\mathsf{T}\mathbf{z} < 0$.

(c) **(5 points)** Assume that the following LP has optimal value OPT.

$$\begin{array}{lrcl} \text{maximize} & x_2 & & \\ \text{subject to} & -x_1 + 2x_2 & \leq & 4 \\ & x_1 & \leq & 3 \\ & x_1 + x_2 & \leq & 5 \\ & x_1, x_2 & \geq & 0, \end{array} \tag{1}$$

What is the optimal value of the following LP? Justify your answer.

$$\begin{array}{lrcl} \text{minimize} & 4y_1 + 5y_2 + 3y_3 & & \\ \text{subject to} & -y_1 + y_2 + y_3 & \geq & 0 \\ & 2y_1 + y_2 & \geq & 1 \\ & y_1, y_2, y_3 & \geq & 0, \end{array} \tag{2}$$