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Approximation Algorithms and Semidefinite Programming FS12
Homework 1

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Due date: March 16, 2012, 10h15.

- Solutions are to be handed in typed in LaTeX. A tutorial can be found at `http://www.cadmo.ethz.ch/education/thesis/latex`.
- Please bring a print-out of your solution with you to the lecture. If you cannot attend, you may alternatively send your solution as a PDF to `stich@inf.ethz.ch`.
- Proofs are to be formally correct, complete and clearly explained.

Exercise 1 (Rank-constrained programs) *(5 Points)*

[Exercise 2.4] A *rank-constrained* semidefinite program is a problem of the form

$$\begin{array}{rl} \max & C \bullet X \\ \text{s.t.} & A(X) = \mathbf{b} \\ & X \succeq 0 \\ & \operatorname{Rank}(X) = k\,, \end{array}$$

where k is a fixed integer. Show that the problem of solving a rank-constrained semidefinite program is NP-hard for $k = 1$.

Exercise 2 (Approximating Max-2-Sat) *(10 Points)*

Let X be a set of variables. A *literal* is any variable $x \in X$ or its negation $\bar{x}$. A *k-clause* is a disjunction of k literals and a *k-CNF formula* is a conjunction of k-clauses. A clause is satisfied by an assignment $f : X \mapsto \{0, 1\}$ if it evaluates to 1 (*true*) under this assignment.

Max-2-Sat is the following computational problem: Given a 2-CNF formula in n variables, determine the maximal number of clauses that can be simultaneously satisfied.

a) Show that we get an 3/4-approximation algorithm by assigning each variable to 0 or 1 with probability $p = 1/2$ each, independent of the other variables.

b) Find a polynomial 0.878-approximation algorithm for Max-2-Sat.

Exercise 3 (Optimization over polynomials) *(10 Points)*

[Exercise 2.7]

a) Let $p(x) \in \mathbb{R}[x]$ be a univariate polynomial of degree d with real coefficients. We would like to decide whether $p(x)$ is a *sum of squares*, i.e., if it can be written as $p(x) = q_1(x)^2 + \cdots + q_m(x)^2$ for some $q_1(x), \ldots, q_m(x) \in \mathbb{R}[x]$. Formulate this problem as the feasibility of a semidefinite program.

b) Let us call a polynomial $p(x) \in \mathbb{R}[x]$ *nonnegative* if $p(x) \geq 0$ for all $x \in R$. Obviously, a sum of squares is nonnegative. Prove that the converse holds as well.

c) Let $p(x) \in \mathbb{R}[x]$ be a given polynomial. Express its global minimum $\min\{p(x) \mid x \in \mathbb{R}\}$ as the optimum of a suitable semidefinite program.

Hint - Exercise 2: Proceed similarly as in the MaxCut approximation. Formulate the problem as an equivalent integer quadratic program. Introduce a variable y_i for each variable x_i of the Max-2-Sat problem and introduce an additional variable y_0 to encode *true* and *false*.

Hint - Exercise 3.a) Find suitable $\mathbf{z}$ and A, s.t. $p(x) = \mathbf{z}^T A \mathbf{z}$.

Hint - Exercise 3.b) Factor into quadratic polynomials.