Randomized Incremental Construction (RIC)

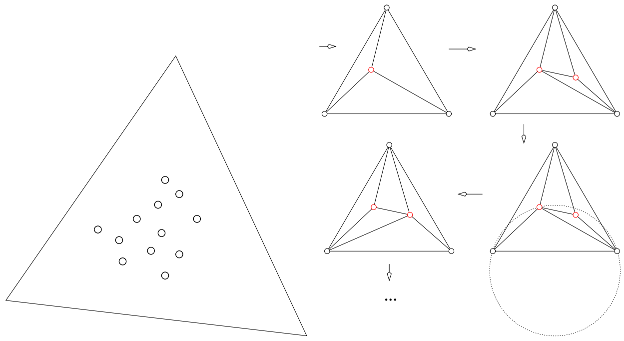
Delaunay Triangulation, Convex Hull in Space, and an Abstract Framework

DT - Incremental Construction (I)

Add points incrementally while maintaining the Delaunay triangulation.

Step 1: Start with a very large triangle containing all points near its center.

Step 2: Add a next point p by placing* it in its triangle—with edges to its three vertices. Then perfom Lawson flips as long as possible. (New edges must be incident to new point p.)

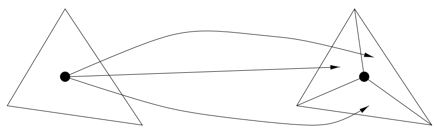


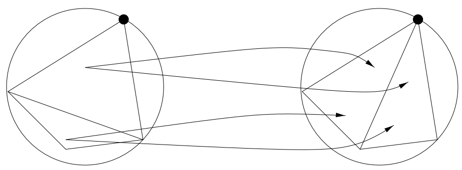
*We assume that no point falls on an edge!

DT - Incremental Construction (II)

How to find the triangle containing p?

History Graph. Every triangle (that ever appears in the process) has pointers (at most three) to the triangles by which it gets covered.





To locate p, trace history pointers.

RIC — Analysis

The number of flips necessary for adding the rth point p_r is $\deg(p_r, \mathcal{T}_r) - 3$, where $\deg(p_r, \mathcal{T}_r)$ denotes the degree of p_r in the resulting triangulation $\mathcal{T}_r$.

There are $3(r+3)-6$ edges in $\mathcal{T}_r$, (3 of which form the large triangle). Hence

$$\sum_{i=1}^{r} \deg(p_i, \mathcal{T}_r) \leq 2(3(r+3)-9) = 6r \ .$$

Since p_r is a uniformly random point in $\{p_1, p_2, \ldots, p_r\}$, its expected degree is bounded by 6.

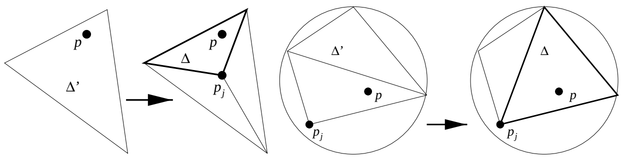
$\longrightarrow$ expected # of flips bounded by $3n$.

We'll see: expected time for locating the points through the history graph is $O(n \log n)$.

Analysis of History Search (I)

Assume $p = p_r$ runs through $\Delta \neq \Delta_0$ (the initial large triangle). Then there is $j \leq r$ such that

- Δ is child of some $\Delta' \in T_{j-1} \setminus T_j$, $p \in$ circumcircle(Δ').



- different Δ have different Δ' (p is in unique successor of Δ')

$\Rightarrow$: length of history path to p

$$\leq 1 + \underbrace{\sum_{j=1}^{r} \sum_{\Delta \in T_{j-1} \setminus T_j} [p \in \text{circumcircle}(\Delta)]}_{S_p}.$$

Analysis of History Search (II)

- Expected overall searchtime:

$$O(n + E(\sum_{p \in P} S_p)).$$

- $T_r, \ldots, T_n$ only contain triangles with $p \notin$ circumcircle(Δ), may let sum run to n:

$$S_p = \sum_{j=1}^{n} \sum_{\Delta \in T_{j-1} \setminus T_j} [p \in \text{circumcircle}(\Delta)]$$

- Only triangles can be destroyed that have been created before:

$$S_p \leq \sum_{j=1}^{n} \sum_{\Delta \in T_j \setminus T_{j-1}} [p \in \text{circumcircle}(\Delta)]$$

Analysis of History Search (III)

The time to search for point p in the history is proportional to (one plus)

$$S_p \leq \sum_{j=1}^{n} \sum_{\Delta \in T_j \setminus T_{j-1}} [p \in \text{circumcircle}(\Delta)]$$

- $p \in$ circumcircle(Δ) $\Leftrightarrow$ (p, Δ) a "conflict"

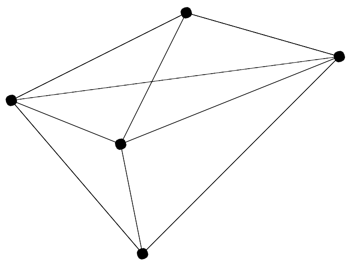
- expected time for all history searches is proportional to (n plus) the expected number of conflicts that appear during the algorithm.

What is this expected number ??? Be patient!

Convex Hull in 3-space

- *Input:* $P \subseteq \mathbb{R}^3, |P| = n$.

- *Output:* (Facets of) the convex hull of P.



Assumption: no four points on a common plane ($\Rightarrow$ all facets are triangles)

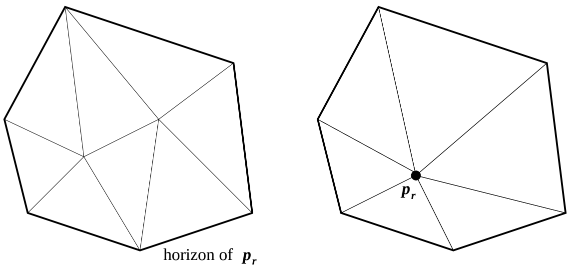
Randomized Incremental Construction

1. Compute convex hull of $\{p_1, \ldots, p_4\} \to C_4$

2. Add points $p_r \in P \setminus \{p_1, \ldots, p_4\}$ in random order:

   - find (and remove) all facets visible from p_r

   - Connect p_r with all its "horizon" vertices $\to C_r$



RIC – Analysis

Step r (adding p_r): the number of new facets is $\deg(p_r, C_r)$.

C_r has at most $3r - 6$ edges, so

$$\sum_{i=5}^{r} \deg(p_i, C_r) \leq 2(3r - 6) < 6r.$$

Since p_r is a uniformly random point in $\{p_5, \ldots, p_r\}$, its expected degree (and therefore the expected number of facets created) is at most 6.

$\Rightarrow$ Overall expected number of facets created (removed) is bounded by $6n$.

Analysis visible facet management (I)

How to find the visible facets for p_r?

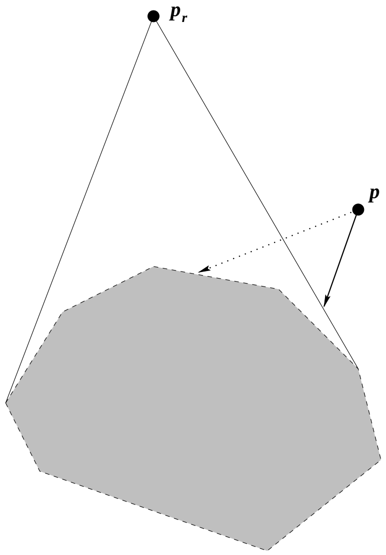
- Maintain for all points $p \notin C_r$ *one* visible facet of C_r, $r = 4, \ldots, n-1$

- From this facet, find all visible facets (and the horizon edges) in time proportional to their number, using depth-first-search.

- in C_4, visible facets for all points can be found in $O(n)$.

- if $p \in P$ loses its visible facet from C_{r-1} to C_r, then either $p \in C_r$, or there exists a new visible facet consisting of p_r and a horizon egde incident to a facet in C_{r-1} that was visible both from p_r and p.

Update of visible facet



Analysis visible facet management (II)

To update p's visible facet in step r, check all (horizon edges of) facets visible both from p and p_r (depth-first search from old visible facet). Throughout this is proportional to (one plus)

$$\begin{aligned} U_p &:= \sum_{r=5}^{n} \sum_{\Delta \in C_{r-1} \setminus C_r} [\Delta \text{ visible from } p] \\ &\leq \sum_{r=5}^{n} \sum_{\Delta \in C_r \setminus C_{r-1}} [\Delta \text{ visible from } p] \end{aligned}$$

- Δ visible from $p \Leftrightarrow (p, \Delta)$ a "conflict"
- expected time to update all visible facets is proportional to (n plus) the expected number of conflicts that appear during the algorithm.

What is this expected number??? Be patient!

An abstract framework

- X a finite set (e.g. set of points P in $\mathbb{R}^2, \mathbb{R}^3$)
- Π a set of *configurations* (e.g. (oriented) triangles defined by three points of P)

Each configuration $\Delta \in \Pi$ has a *defining set*

$$D(\Delta) \subseteq X$$

(e.g. the vertices of the triangle) and a *conflict set*

$$K(\Delta) \subseteq X \quad \text{("killers")}$$

(e.g. points in the circumcircle of the triangle (P in $\mathbb{R}^2$) or points that are visible from the triangle (P in $\mathbb{R}^3$) – here we need orientation).

Properties we need

- $D(\Delta) \leq d$, for all $\Delta \in \Pi$
- $D(\Delta) \cap K(\Delta) = \emptyset$, for all $\Delta \in \Pi$
- Only constantly many configurations have the same defining set (technical condition)

Definitions

- (X, Π, D, K) is a *configuration space* of dimension d
- For $R \subseteq X$,

  $\mathcal{T}(R) := \{\Delta \in \Pi \mid D(\Delta) \subseteq R, K(\Delta) \cap R = \emptyset\}$

  is the set of *active configurations* with respect to R.

Final Goal

Compute the active configurations w.r.t. X,

$$\mathcal{T}(X) = \{\Delta \in \Pi \mid K(\Delta) = \emptyset\}$$

(e.g. all Delaunay triangles (P in $\mathbb{R}^2$), all facets of the convex hull (P in $\mathbb{R}^3$))

Algorithm

- Randomized incremental: add elements of X in random order, maintain

  $\mathcal{T}_r$:= set of active configurations w.r.t. first r elements $\{x_1, \ldots, x_r\}$

RIC – Analysis

The number of new configurations created when adding element x_r is equal to $\text{deg}(x_r, \mathcal{T}_r)$, the number of configurations in $\mathcal{T}_r$ that have x_r in its defining set. Because each configuration has at most d defining elements, we have

$$\sum_{i=1}^{r} \text{deg}(x_r, \mathcal{T}_r) \leq d|\mathcal{T}_r|.$$

Since x_r is uniformly random in $\{x_1, \ldots, x_r\}$, its expected degree is bounded by

$$\frac{d}{r}|\mathcal{T}(R)|,$$

for any fixed $R = \{x_1, \ldots, x_r\}$. Averaging over R it follows that the expected number of new configurations is bounded by

$$\frac{d}{r}\underbrace{E(|\mathcal{T}_r|)}_{t_r}.$$

Expected number of conflicts

We want to count the overall number of conflicts that appear during the algorithms, i.e.

$$\sum_{r=1}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|.$$

The following are equal: the conflicts

- appearing from $\mathcal{T}_{r-1}$ to $\mathcal{T}_r$,
- involving configurations $\Delta \in \mathcal{T}_r$ with $x_r \in D(\Delta)$.

For fixed $R = \{x_1, \ldots, x_r\}$, $\text{prob}(x = x_r) = 1/r$ for $x \in R$, so the expected conflict number is

$$\begin{aligned} & \frac{1}{r} \sum_{x \in R} \sum_{\Delta \in \mathcal{T}(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)] \\ \leq \; & \frac{d}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|. \end{aligned}$$

A crucial Lemma

Lemma.

$$\begin{aligned} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\} \;=\; & |\mathcal{T}(R)| - |\mathcal{T}(R \cup \{y\})| \\ & + \text{deg}(y, \mathcal{T}(R \cup \{y\})). \end{aligned}$$

Proof. The configurations of $\mathcal{T}(R)$ not in conflict with y are exactly the configurations of $\mathcal{T}(R \cup \{y\})$ that do not have y in their defining set.

Expected number of conflicts (II)

K_r: expected number of new conflicts when x_r is inserted.

$$\begin{aligned} K_r \;\leq\; & \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R| = r} \frac{d}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}| \\ =\; & \underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R| = r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)|}_{k_1} - \\ & \underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R| = r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})|}_{k_2} + \\ & \underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R| = r} \frac{d}{r} \sum_{y \in X \setminus R} \text{deg}(y, \mathcal{T}(R \cup \{y\}))}_{k_3}. \end{aligned}$$

Evaluating k_1

$$\begin{aligned} k_1 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} \frac{d}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R)| \\ &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} |\mathcal{T}(R)| \frac{d}{r} \sum_{y\in X\setminus R} 1 \\ &= \frac{d}{r}(n-r)t_r. \end{aligned}$$

Evaluating k_2

$$\begin{aligned} k_2 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} \frac{d}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R\cup\{y\})| \\ &= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} \sum_{y\in R'} |\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d}{r}(r+1)|\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r}(n-r)|\mathcal{T}(R')| \\ &= \frac{d}{r}(n-r)t_{r+1} \\ &= \frac{d}{r+1}(n-(r+1))t_{r+1} + \frac{dn}{r(r+1)}t_{r+1}. \end{aligned}$$

Evaluating k_3

$$\begin{aligned} k_3 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} \frac{d}{r} \sum_{y\in X\setminus R} \mathrm{deg}(y, \mathcal{T}(R\cup\{y\})) \\ &= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} \sum_{y\in R'} \mathrm{deg}(y, \mathcal{T}(R')) \\ &\leq \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} d|\mathcal{T}(R')| \\ &= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d}{r} d|\mathcal{T}(R')| \\ &\leq \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{n-r}{r+1} \cdot \frac{d}{r} d|\mathcal{T}(R')| \\ &= \frac{d^2}{r(r+1)}(n-r)t_{r+1} \\ &= \frac{d^2 n}{r(r+1)}t_{r+1} - \frac{d^2}{r+1}t_{r+1}. \end{aligned}$$

Expected number of conflicts (III)

In step n, no conflict is created. Moreover, $k_1(r+1)$ cancels with the first term of $k_2(r)$, and we get

$$\begin{aligned} \sum_{r=1}^{n-1} K_r &\leq \sum_{r=1}^{n-1} (k_1 - k_2 + k_3) \\ &\leq d(n-1)t_1 + \\ &\quad d(d-1)n \sum_{r=1}^{n-1} \frac{t_{r+1}}{r(r+1)} - \\ &\quad d^2 \sum_{r=1}^{n-1} \frac{t_{r+1}}{r+1}. \end{aligned}$$

Examples: Delaunay Triangulation and Convex Hull in 3-space

- $d = 3$
- $t_r \leq 2r - 5 = O(r)$
- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

Theorem: The Delaunay-Triangulation of n points in the plane and the convex hull of n points in space can be computed in expected time $O(n \log n)$ (in case of general position).

Example: Convex Hull in 2-space

- $d = 2$
- $t_r \leq r = O(r)$
- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

If $t_r = o(r) \Rightarrow O(n)$.