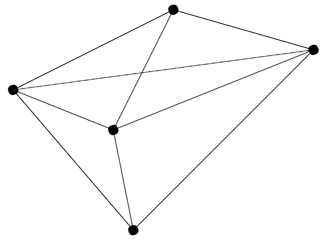
Randomized Incremental Construction (RIC)

Convex Hulls in Space, and an Abstract Framework

Convex Hull in 3-space

The convex hull of n points in $\mathbb{R}^3$ is a *convex polytope* in $\mathbb{R}^3$.

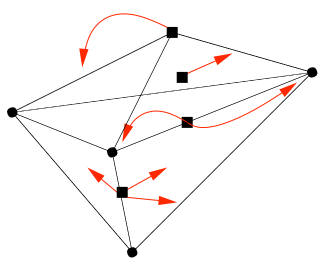


The vertices and edges form a planar graph with at most $3n - 6$ edges and at most $2n - 4$ facets (Steinitz's Theorem, Euler formula).

Assumption: no four points are on a common plane $\Rightarrow$ all *facets* of the convex hull are triangles (assumption can be removed...)

Convex Hull Computation in 3-space

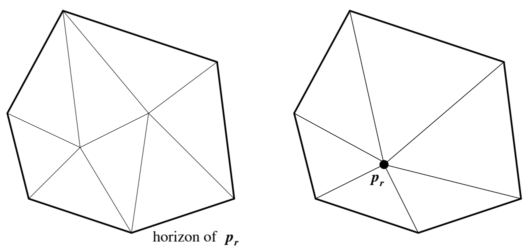
- *Input:* $P \subseteq \mathbb{R}^3, |P| = n$.
- *Output:* The planar graph of vertices, edges, and facets of conv(P) (suitably linked via DCEL).



- algorithm can compute *facet graph* in any dimension d

Randomized Incremental Construction

1. Compute convex hull of $\{p_1, \ldots, p_4\} \rightarrow C_4$
2. Add points $p_r \in P \setminus \{p_1, \ldots, p_4\}$ in random order:
   - find (and remove) all facets visible from p_r
   - Connect p_r with all its "horizon" vertices $\rightarrow C_r$



RIC – Analysis

Step r (adding p_r): the number of new facets is $\deg(p_r, C_r)$.

C_r has at most $3r-6$ edges, so

$$\sum_{p\in\{p_5,\ldots,p_r\}} \deg(p, C_r) \le 2(3r-6) < 6r.$$

Since p_r is a random point in $\{p_5,\ldots,p_r\}$, its expected degree (and therefore the expected number of facets created) is at most

$$\frac{1}{r-4}\sum_{p\in\{p_5,\ldots,p_r\}} \deg(p, C_r) \approx 6.$$

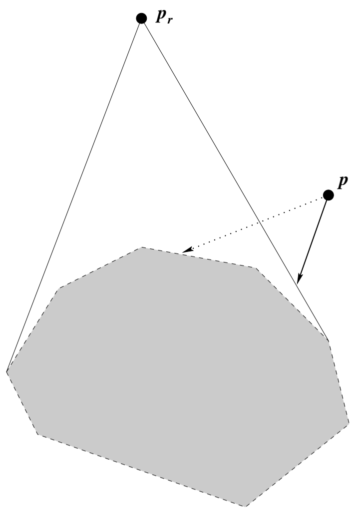
$\Rightarrow$ Overall expected number of facets created (removed) is bounded by $\approx 6n$.

Analysis visible facet management (I)

How to find the visible facets for p_r?

- Maintain for all points $p \notin C_r$ *one* visible facet of C_r, $r = 4, \ldots, n-1$
- From this facet, find all visible facets (and the horizon edges) in time proportional to their number, using depth-first-search.
- in C_4, visible facets for all points can be found in $O(n)$.
- if $p \in P$ loses its visible facet from C_{r-1} to C_r, then either $p \in C_r$, or there exists a new visible facet consisting of p_r and a horizon egde incident to a facet in C_{r-1} that was visible both from p_r and p.

Update of visible facet



Analysis visible facet management (II)

To update p's visible facet in step r, check all (horizon edges of) facets visible both from p and p_r (depth-first search from old visible facet). Throughout this is proportional to (one plus)

$$\begin{aligned} U_p &:= \sum_{r=5}^{n} \sum_{\Delta\in C_{r-1}\setminus C_r} [\Delta \text{ visible from } p] \\ &\le \sum_{r=5}^{n} \sum_{\Delta\in C_r\setminus C_{r-1}} [\Delta \text{ visible from } p] \end{aligned}$$

- Δ visible from $p \Leftrightarrow (p, \Delta)$ a "conflict"
- expected time to update all visible facets is proportional to (n plus) the expected number of conflicts that appear during the algorithm.

What is this expected number??? Be patient!

An abstract framework

- X a finite set (e.g. set of points P in $\mathbb{R}^2, \mathbb{R}^3$)

- Π a set of *configurations* (e.g. oriented triangles defined by three points of P)

Each configuration $\Delta \in \Pi$ has a *defining set*

$$D(\Delta) \subseteq X$$

(e.g. the vertices of the triangle) and a *conflict set*

$$K(\Delta) \subseteq X \quad (\text{“killers”})$$

(e.g. points from which the triangle is visible – here we need orientation).

Properties we need

- $D(\Delta) \leq d$, for all $\Delta \in \Pi$

- $D(\Delta) \cap K(\Delta) = \emptyset$, for all $\Delta \in \Pi$

- Only constantly many configurations have the same defining set (technical condition)

Definitions

- (X, Π, D, K) is a *configuration space* of dimension d

- For $R \subseteq X$,

  $\mathcal{T}(R) := \{\Delta \in \Pi \mid D(\Delta) \subseteq R, K(\Delta) \cap R = \emptyset\}$

  is the set of *active configurations* with respect to R.

Final Goal

Compute the active configurations w.r.t. X,

$$\mathcal{T}(X) = \{\Delta \in \Pi \mid K(\Delta) = \emptyset\}$$

(e.g. all facets of the convex hull (P in $\mathbb{R}^3$))

Algorithm

- Randomized incremental: add elements of X in random order, maintain

  $\mathcal{T}_r$:= set of active configurations w.r.t. first r elements $\{x_1, \ldots, x_r\}$

RIC – Analysis

The number of new configurations created in adding element x_r is equal to deg$(x_r, \mathcal{T}_r)$, the number of configurations in $\mathcal{T}_r$ that have x_r in its defining set. Because each configuration has at most d defining elements, we have

$$\sum_{x \in \{x_1, \ldots, x_r\}} \text{deg}(x, \mathcal{T}_r) \leq d|\mathcal{T}_r|.$$

Since x_r is random in $\{x_1, \ldots, x_r\}$, its expected degree is bounded by

$$\frac{1}{r} \sum_{x \in \{x_1, \ldots, x_r\}} \text{deg}(x, \mathcal{T}_r) \leq \frac{d}{r} |\mathcal{T}(R)|,$$

for any fixed $R = \{x_1, \ldots, x_r\}$. Averaging over R it follows that the expected number of new configurations is bounded by

$$\frac{d}{r} \underbrace{E(|\mathcal{T}_r|)}_{t_r}.$$

Expected number of conflicts

We want to count the overall number of conflicts (x, Δ) that appear during the algorithms, i.e.

$$\sum_{r=1}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|.$$

The following are equal: the conflicts

- appearing in the step $\mathcal{T}_{r-1} \to \mathcal{T}_r$,
- involving some $\Delta \in \mathcal{T}_r$ with $x_r \in D(\Delta)$.

For fixed $R = \{x_1, \ldots, x_r\}$, prob$(x = x_r) = 1/r$ for $x \in R$, so the expected conflict number is

$$\frac{1}{r} \sum_{x \in R} \sum_{\Delta \in \mathcal{T}(R), x \in D(\Delta)} \sum_{y \in X \setminus R} [y \in K(\Delta)]$$
$$\leq \frac{d}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|.$$

An easy but crucial Lemma

Lemma.

$$|\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}$$
$$=$$
$$|\mathcal{T}(R)| - |\mathcal{T}(R \cup \{y\})| + \text{deg}(y, \mathcal{T}(R \cup \{y\})).$$

Proof. The configurations of $\mathcal{T}(R)$ not in conflict with y are exactly the configurations of $\mathcal{T}(R \cup \{y\})$ that do not have y in their defining set.

Expected number of conflicts (II)

K_r: expected number of new conflicts when x_r is inserted. K_r is bounded by

$$\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) \mid y \in K(\Delta)\}|$$

which is

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)|}_{k_1} -$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})|}_{k_2} +$$

$$\underbrace{\frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} \text{deg}(y, \mathcal{T}(R \cup \{y\}))}_{k_3}.$$

Evaluating k_1

$$\begin{aligned} k_1 &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)| \\ &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} |\mathcal{T}(R)| \frac{d}{r} \sum_{y \in X \setminus R} 1 \\ &= \frac{d}{r}(n-r)t_r. \end{aligned}$$

Evaluating k_2

$$
\begin{aligned}
k_2 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} \frac{d}{r} \sum_{y\in X\setminus R} |\mathcal{T}(R\cup\{y\})| \\
&= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} \sum_{y\in R'} |\mathcal{T}(R')| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d}{r} (r+1)|\mathcal{T}(R')| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r}(n-r)|\mathcal{T}(R')| \\
&= \frac{d}{r}(n-r)t_{r+1} \\
&= \frac{d}{r+1}(n-(r+1))t_{r+1} + \frac{dn}{r(r+1)}t_{r+1}.
\end{aligned}
$$

Evaluating k_3

$$
\begin{aligned}
k_3 &= \frac{1}{\binom{n}{r}} \sum_{R\subseteq X, |R|=r} \frac{d}{r} \sum_{y\in X\setminus R} \mathrm{deg}(y, \mathcal{T}(R\cup\{y\})) \\
&= \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} \sum_{y\in R'} \mathrm{deg}(y, \mathcal{T}(R')) \\
&\leq \frac{1}{\binom{n}{r}} \sum_{R'\subseteq X, |R'|=r+1} \frac{d}{r} d|\mathcal{T}(R')| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d}{r} d|\mathcal{T}(R')| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{R'\subseteq X, |R'|=r+1} \frac{n-r}{r+1} \cdot \frac{d}{r} d|\mathcal{T}(R')| \\
&= \frac{d^2}{r(r+1)}(n-r)t_{r+1} \\
&= \frac{d^2 n}{r(r+1)}t_{r+1} - \frac{d^2}{r+1}t_{r+1}.
\end{aligned}
$$

Expected number of conflicts (III)

In step n, no conflict is created. Moreover, $k_1(r+1)$ cancels with the first term of $k_2(r)$, and we get

$$
\begin{aligned}
\sum_{r=1}^{n-1} K_r &\leq \sum_{r=1}^{n-1} (k_1 - k_2 + k_3) \\
&\leq d(n-1)t_1 + \\
&\quad d(d-1)n \sum_{r=1}^{n-1} \frac{t_{r+1}}{r(r+1)} - \\
&\quad d^2 \sum_{r=1}^{n-1} \frac{t_{r+1}}{r+1}.
\end{aligned}
$$

Example: Convex Hull in 3-space

- $d = 3$
- $t_r \leq 2r - 4 = O(r)$
- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

Theorem: The convex hull of n points in 3-space can be computed in expected time

$$O(n \log n).$$

Corollary: A Delaunay triangulation of n points in 2-space can be computed in expected time

$$O(n \log n).$$

Example: Convex Hull in d-space

- $t_r = O(r^{\lfloor d/2 \rfloor})$

- $\sum_{r=1}^{n-1} K_r = O(n^{\lfloor d/2 \rfloor})$

This is *worst-case-optimal*, since there are sets of n points whose convex hull has $\Theta(n^{\lfloor d/2 \rfloor})$ facets (Mc Mullen's Upper Bound Theorem).

Example: Convex Hull in 2-space

- $d = 2$

- $t_r \leq r = O(r)$

- $\sum_{r=1}^{n-1} K_r = O(n + nH_{n-1}) \Rightarrow O(n \log n)$.

If $t_r = o(r) \Rightarrow O(n)$. This happens for example when the n points are chosen randomly from the unit square or the unit disk.