1. Planar Convex Hull

Lecture on Monday 22nd September, 2008 by Michael Hoffmann <hoffmann@inf.ethz.ch>

1.1 Basic Geometric Objects

We will mostly be concerned with the d-dimensional Euclidean space $\mathbb{R}^d$, for small $d \in \mathbb{N}$; typically, $d = 2$ or $d = 3$. The basic objects of interest in $\mathbb{R}^d$ are the following.

Points. A point p, typically described by its d Cartesian coordinates $p = (x_1, \ldots, x_d)$.

Directions. A vector $v \in \mathcal{S}^{d-1}$ (the $(d-1)$-dimensional unit sphere), typically described by its d Cartesian coordinates $v = (x_1, \ldots, x_d)$.

Lines. A line is a one-dimensional affine subspace. It can be described by a point p and a direction d as the set of all points r that satisfy $r = p + \lambda d$, for some $\lambda \in \mathbb{R}$.

Rays. A ray is a connected component of what remains if one removes a single point from a line. It can be described by a point p and a direction d as the set of all points r that satisfy $r = p + \lambda d$, for some $\lambda \geq 0$.

Line segment. A line segment is the bounded connected component of what remains if one removes a single point from a ray. It can be described by two points p and q as the set of all points r that satisfy $r = p + \lambda(q - p)$, for some $\lambda \in [0, 1]$. We will denote the line segment through p and q by $\overline{pq}$.

Hyperplanes. A hyperplane $\mathcal{H}$ is a $(d-1)$-dimensional affine subspace. It can be described algebraically by $d + 1$ coefficients $\lambda_1, \ldots, \lambda_{d+1} \in \mathbb{R}$ as the set of all points $(x_1, \ldots, x_d)$ that satisfy the linear equation $\mathcal{H} : \sum_{i=1}^{d} \lambda_i x_i = \lambda_{d+1}$.

Spheres. A sphere is the set of all points that are equidistant to a fixed point. It can be described by a point c (center) and a number $\rho \in \mathbb{R}$ (radius) as the set of all points p that satisfy $||p - c|| \leq \rho$.

1.2 Convexity

Consider $P \subset \mathbb{R}^d$. The following terminology should be familiar from linear algebra courses.

Linear hull.

$$\mathrm{lin}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \forall\, i : p_i \in P, \lambda_i \in \mathbb{R} \right\}$$

(smallest linear subspace containing P). For instance, if $P = \{p\} \subset \mathbb{R}^2 \setminus \{0\}$ then $\mathrm{lin}(P)$ is the line through p and the origin.

Affine hull.

$$\mathrm{aff}(P) := \left\{ q \;\middle|\; q = \sum \lambda_i p_i \wedge \sum \lambda_i = 1 \wedge \forall\, i : p_i \in P, \lambda_i \in \mathbb{R} \right\}$$

(smallest affine subspace containing P). For instance, if $P = \{p, q\} \subset \mathbb{R}^2$ and $p \neq q$ then $\mathrm{aff}(P)$ is the line through p and q.

Convex hull.

Definition 1.1 *A set $P \subseteq \mathbb{R}^d$ is* **convex** *if and only if $\overline{pq} \subseteq P$, for any $p, q \in P$.*

Theorem 1.2 *A set $P \subseteq \mathbb{R}^d$ is convex if and only if $\sum_{i=1}^{n} \lambda_i p_i \in P$, for all $n \in \mathbb{N}$, $p_1, \ldots, p_n \in P$, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.*

Proof. "$\Leftarrow$": obvious with $n = 2$.

"$\Rightarrow$": Induction on n. For $n = 1$ the statement is trivial. For $n \geq 2$, let $p_i \in P$ and $\lambda_i \geq 0$, for $1 \leq i \leq n$, and assume $\sum_{i=1}^{n} \lambda_i = 1$. We may suppose that $\lambda_i > 0$, for all i. (Simply omit those points whose coefficient is zero.)

Define $\lambda = \sum_{i=1}^{n-1} \lambda_i$ and for $1 \leq i \leq n-1$ set $\mu_i = \lambda_i / \lambda$. Observe that $\mu_i \geq 0$ and $\sum_{i=1}^{n-1} \mu_i = 1$. By the inductive hypothesis, $q := \sum_{i=1}^{n-1} \mu_i p_i \in P$, and thus by convexity of P also $\lambda q + (1 - \lambda) p_k \in P$. We conclude by noting that $\lambda q + (1 - \lambda) p_k = \lambda \sum_{i=1}^{n-1} \mu_i p_i + \lambda_k p_k = \sum_{i=1}^{n} \lambda_i p_i$. □

Observation 1.3 *For any family $(P_i)_{i \in I}$ of convex sets the intersection $\bigcap_{i \in I} P_i$ is convex.*

Definition 1.4 *The* **convex hull** $\mathrm{conv}(P)$ *of a set $P \subset \mathbb{R}^d$ is the intersection of all convex supersets of P.*

By Observation 1.3, the convex hull is convex, indeed.

Theorem 1.5 *For any $P \subseteq \mathbb{R}^d$ we have*

$$\mathrm{conv}(P) = \left\{ \sum_{i=1}^{n} \lambda_i p_i \;\middle|\; n \in \mathbb{N} \;\wedge\; \sum_{i=1}^{n} \lambda_i = 1 \;\wedge\; \forall i \in \{1, \ldots, n\} : \lambda_i \geq 0 \wedge p_i \in P \right\}.$$

Proof. "$\supseteq$": Consider a convex set $C \supseteq P$. By Theorem 1.2 the right hand side is contained in C. As C was arbitrary, the claim follows.

"$\subseteq$": We show that the right hand side forms a convex set. Let $p = \sum_{i=1}^{n} \lambda_i p_i$ and $q = \sum_{i=1}^{n} \mu_i p_i$ be two convex combinations. (We may suppose that both p and q are expressed over the same p_i by possibly adding some terms with a coefficient of zero.)

Then for $\lambda \in [0,1]$ we have $\lambda p + (1-\lambda)q = \sum_{i=1}^{n}(\lambda\lambda_i + (1-\lambda)\mu_i)p_i \in P$, as $\sum_{i=1}^{n}(\lambda\lambda_i + (1-\lambda)\mu_i) = \lambda + (1-\lambda) = 1$. □

Definition 1.6 *The convex hull of a finite point set* $P \subset \mathbb{R}^d$ *forms a* **convex polytope**. *Each* $p \in P$ *for which* $p \notin \text{conv}(P \setminus \{p\})$ *is called a* **vertex** *of* $\text{conv}(P)$.

Essentially, the following theorem shows that the term vertex above is well defined.

Theorem 1.7 *A convex polytope in* $\mathbb{R}^d$ *is the convex hull of its vertices.*

Proof. Let $P = \text{conv}(p_1, \ldots, p_n)$, $n \in \mathbb{N}$, such that without loss of generality $p_1, \ldots, p_k$ are the vertices of $\mathcal{P} := \text{conv}(P)$. We prove by induction on n that $\text{conv}(p_1, \ldots, p_n) \subseteq \text{conv}(p_1, \ldots, p_k)$. For $n = k$ the statement is trivial.

For $n > k$, p_n is not a vertex of $\mathcal{P}$ and hence p_n can be expressed as a convex combination $p_n = \sum_{i=1}^{n-1} \lambda_i p_i$. Thus for any $x \in \mathcal{P}$ we can write $x = \sum_{i=1}^{n} \mu_i p_i = \sum_{i=1}^{n-1} \mu_i p_i + \mu_k \sum_{i=1}^{n-1} \lambda_i p_i = \sum_{i=1}^{n-1}(\mu_i + \mu_k\lambda_i)p_i$. As $\sum_{i=1}^{n-1}(\mu_i + \mu_k\lambda_i) = 1$, we conclude by the inductive hypothesis that $x \in \text{conv}(p_1, \ldots, p_k)$. □

Theorem 1.8 (Carathéodory) *For any* $P \subset \mathbb{R}^d$ *and* $q \in \text{conv}(P)$ *there exist* $k \leq d+1$ *points* $p_1, \ldots, p_k \in P$ *such that* $q \in \text{conv}(p_1, \ldots, p_k)$.

Theorem 1.9 *For* $P \subset \mathbb{R}^d$ *we can characterize* $\text{conv}(P)$ *equivalently as one of*

a) the smallest convex subset of $\mathbb{R}^d$ *that contains* P;

b) the intersection of all convex supersets of P;

c) the intersection of all closed halfplanes containing P.

[McMullen-Shephard 1971]

1.3 Models of Computation

Real RAM Model. A memory cell stores a real number. Any single arithmetic operation or comparison can be computed in constant time. In addition, sometimes also roots, logarithms, other analytic functions, indirect addressing (integral), or floor and ceiling are used.

This is a quite powerful (and somewhat unrealistic) model of computation. Therefore we have to ensure that we do not abuse its power.

Algebraic Computation Trees [Ben-Or 1983]. A computation is regarded a binary tree.

- The leaves contain the (possible) results of the computation.
- Every node ν with one child an operation of the form $+, -, *, /, \sqrt{}, \dots$ is associated to. The operands of this operation are constant, input values, or among the ancestors of ν in the tree.
- Every node ν with two children a branching of the form > 0, ≥ 0, or $= 0$ is associated to. The branch is with respect to the result of ν's parent node. If the expression yields `true`, the computation continues with the left child of ν; otherwise, it continues with the right child of ν.

If every branch is based on a linear function in the input values, we face a *linear computation tree*. Analogously one can define, for instance, quadratic computation trees.

The term *decision tree* is used if all of the results are either `true` or `false`.

1.4 The convex hull problem in $\mathbb{R}^2$

Convex hull

Input: $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Sequence $(q_1, \dots, q_h)$, $1 \leq h \leq n$, of the vertices of $\text{conv}(P)$ (ordered counterclockwise).

Extremal points

Input: $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$.

Output: Set $Q \subseteq P$ of the vertices of $\text{conv}(P)$.

Degeneracies. Three points collinear. Which are extremal?

Definition 1.10 *A point* $p \in P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ *is* **extremal** *for* $P \iff$ *there is a directed line* g *through* p *such that* $P \setminus \{p\}$ *is to the left of* g.

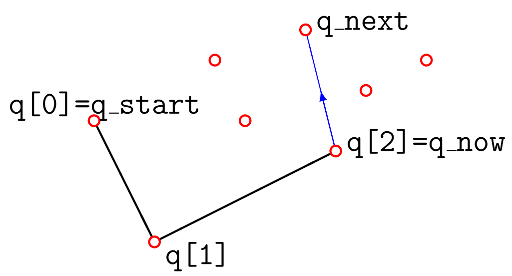
1.5 Trivial algorithms

Test for every point $p \in P$ whether there are $q, r, s \in P \setminus \{p\}$ such that p is inside the triangle with vertices q, r, and s. Runtime $O(n^4)$.

Test for every pair $(p, q) \in P^2$ whether all points from $P \setminus \{p, q\}$ are to the left of the directed line through p and q (or on the line segment $\overline{pq}$). Runtime $O(n^3)$.

1.6 Jarvis' Wrap

Find a point p_1 that is a vertex of $\text{conv}(P)$ (e.g., the one with smallest x-coordinate). "Wrap" P starting from p_1, i.e., always find the next vertex of $\text{conv}(P)$ as the one that is rightmost with respect to the previous vertex.



Analysis. For every output point n rightturn tests, that is, $\Rightarrow O(nh)$. (Worst case $h = n$, i.e., $O(n^2)$.)

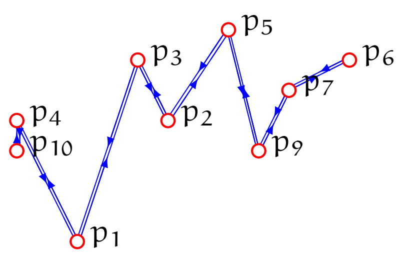
Jarvis' Wrap has a remarkable property that is called *output dependence*: the runtime depends not only on the size of the input but also on the size of the output. For a huge point set it constructs the convex hull in optimal linear time, if the convex hull consists of a constant number of vertices only. Unfortunately the worst case performance of Jarvis' Wrap is suboptimal, as we will see soon.

Degeneracies.

- Several points have smallest x-coordinate $\Rightarrow$ lexicographic order: $(p_x, p_y) < (q_x, q_y) \iff p_x < q_x \vee p_x = q_x \wedge p_y < q_y$.
- Several points identical.
- Three or more points collinear $\Rightarrow$ choose the point that is farthest among those that are rightmost.

1.7 Graham Scan (SLR)

Sort points lexicographically and remove duplicates: $(p_1, \ldots, p_n)$.



$$p_{10}\, p_4\, p_1\, p_3\, p_2\, p_5\, p_9\, p_7\, p_6\, p_7\, p_9\, p_5\, p_2\, p_3\, p_1\, p_4\, p_{10}$$

As long as there is a (consecutive) triple (p, q, r) s.t. q is left of or on the directed line $\overrightarrow{pr}$, remove q from the sequence.

Analysis.

1. Sorting and removal of duplicate points: $O(n \log n)$.
2. At begin: $2n - 2$ points; at the end: h points. $\Rightarrow 2n - h - 2$ shortcuts/positive rightturn tests. In addition at most $2n - 2$ negative tests. Altogether at most $4n - h - 4$ rightturn tests.

In total $O(n \log n)$ time.

There are many variations of this algorithm, the basic idea is due to Graham['72].

References