13. Davenport-Schinzel Sequences

Lecture on Thursday 13^{th} November, 2008 by Michael Hoffmann <hoffmann@inf.ethz.ch>

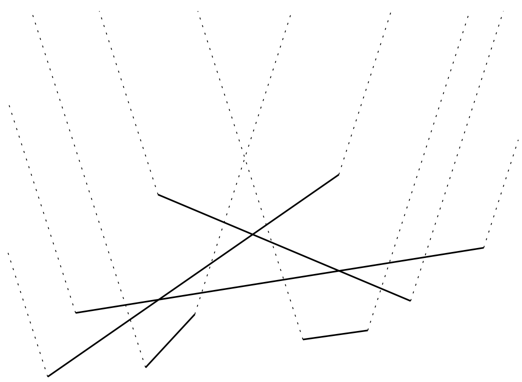
Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a collection of real-valued continuous functions defined on a common interval $I \subset \mathbb{R}$. The *lower envelope* $\mathcal{L}_\mathcal{F}$ of $\mathcal{F}$ is defined as the pointwise minimum of the functions f_i, $1 \leq i \leq n$, over I. Suppose that any pair f_i, f_j, $1 \leq i < j \leq n$, intersects in at most s points. Then I can be decomposed into a finite sequence $I_1, \ldots, I_\ell$ of (maximal connected) pieces on each of which a single function from $\mathcal{F}$ defines $\mathcal{L}_\mathcal{F}$. Define the sequence $\phi(\mathcal{F}) = (\phi_1, \ldots, \phi_\ell)$, where f_{ϕ_i} is the function from $\mathcal{F}$ which defines $\mathcal{L}_\mathcal{F}$ on I_i.

Observation 13.1 *$\phi(\mathcal{F})$ is an (n, s)-Davenport-Schinzel sequence.*

In the case of line segments the above statement does not hold because a set of line segments is in general not defined on a common real interval.

Proposition 13.2 *Let $\mathcal{F}$ be a collection of n real-valued continuous functions each of which is defined on some real interval. If any two functions from $\mathcal{F}$ intersect in at most s points then $\phi(\mathcal{F})$ is an $(n, s+2)$-Davenport-Schinzel sequence.*

Proof. Let I denote the union of all intervals on which one of the functions from $\mathcal{F}$ is defined. Consider any function $f \in \mathcal{F}$ defined on $[a, b] \subseteq I = [c, d]$. Extend f on I by extending it using almost vertical rays pointing upward, from a use a ray of sufficiently small[1] slope, from b use a ray of sufficiently large slope. For all functions use the same slope on these two extensions such that no extensions in the same direction intersect. Denote the resulting collection of functions totally defined on I by $\mathcal{F}'$. If the rays are sufficiently close to vertical then $\phi(\mathcal{F}') = \phi(\mathcal{F})$. (There is a minor complication in case that two defining intervals share an endpoint: If some $f \in \mathcal{F}$ is defined on $[a, b]$ and some $g \in \mathcal{F}$ is defined on $[b, c]$ and without loss of generality $f(b) < g(b)$, then modify the defining interval of f to be $[a, b - \epsilon]$, where ϵ is chosen such that the right extension ray of f passes through the point $(b, g(b))$.)



[1] In particular, such that all intersections of any two rays lie above every function from $\mathcal{F}$.

For any $f \in \mathcal{F}'$ a single extension ray can create at most one additional intersection with any $g \in \mathcal{F}'$. (Let $[a_f, b_f]$ and $[a_g, b_g]$ be the intervals on which the function f and g, respectively, was defined originally. Consider the ray r extending f from a_f to the left. If $a_f \in [a_g, b_g]$ then r may create a new intersection with g, if $a_f > b_g$ then r creates a new intersection with the right extension of g from b_g, and if $a_f < a_g$ then r does not create any new intersection with g.)

On the other hand, for any pair s, t of segments, neither the left extension of the leftmost segment endpoint nor the right extension of the rightmost segment endpoint can introduce an additional intersection. Therefore, any pair of segments in $\mathcal{F}'$ intersects at most $s + 2$ times and the claim follows. □

Next we will give an upper bound on the length of Davenport-Schinzel sequences for small s.

Lemma 13.3 $\lambda_1(n) = n$, $\lambda_2(n) = 2n - 1$, *and* $\lambda_3(n) \leq 2n(1 + \log n)$.

Proof. $\lambda_1(n) = n$ is obvious. $\lambda_2(n) = 2n - 1$ is given as an exercise. We prove $\lambda_3(n) \leq 2n(1 + \log n) = O(n \log n)$.

For $n = 1$ it is $\lambda_3(1) = 1 \leq 2$. For $n > 1$ consider any $(n, 3)$-DS sequence σ of length $\lambda_3(n)$. Let a be a character which appears least frequently in σ. Clearly a appears at most $\lambda_3(n)/n$ times in σ. Delete all appearances of a from σ to obtain a sequence σ' on $n - 1$ symbols. But σ' is not necessarily a DS-sequence because there may be consecutive appearances of a character b in σ', in case that $\sigma = \ldots bab \ldots$.

Claim: There are at most two pairs of consecutive appearances of the same character in σ'. Indeed, such a pair can be created around the first and last appearance of a in σ only. If any intermediate appearance of a creates a pair bb in σ' then $\sigma = \ldots a \ldots bab \ldots a \ldots$, in contradiction to σ being an $(n, 3)$-DS sequence.

Therefore, one can remove at most two characters from σ' to obtain a $(n - 1, 3)$-DS-sequence $\tilde{\sigma}$. As the length of $\tilde{\sigma}$ is bounded by $\lambda_3(n - 1)$, we obtain $\lambda_3(n) \leq \lambda_3(n - 1) + \lambda_3(n)/n + 2$. Reformulating yields

$$\frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n-1)}{n-1} + \frac{2}{n-1} \leq 1 + 2\sum_{i=1}^{n-1} \frac{1}{i} = 1 + 2H_{n-1} < 1 + 2(1 + \log n)$$

and together with $2H_{n-1} < 1 + 2\log n$ we obtain $\lambda_3(n) \leq 2n(1 + \log n)$. □

Remarks. The upper bound is not tight. It can be shown that $\lambda_3(n) = \Theta(n\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann Function. More precisely, Hart and Sharir have shown in 1986 that $\frac{1}{2}n(\alpha(n) - 4) \leq \lambda_3(n) \leq (68n - 32)\alpha(n) + (70n - 32)$.

The Ackermann function is defined on $\mathbb{N} \times \mathbb{N}$ as follows. $A(1, n) = 2n$, $A(k, 1) = 2$, for $k \geq 2$, and $A(k, n) = A(k - 1, A(k, n - 1))$, for $k, n \geq 2$. The inverse Ackermann Function $\alpha(n)$ is then given by $\alpha(n) = \min\{k \in \mathbb{N} \mid A(k, k) \geq n\}$. As $A(4, 4)$ is a tower with 65536 2's, $\alpha(n) \leq 4$ for all practical purposes.

$\lambda_s(n)$ is almost linear even for larger values of s. For example, $\lambda_4(n) = \Theta(n2^{\alpha(n)})$.