8. Voronoi Diagrams

Lecture on Monday 20th October, 2008 by Michael Hoffmann <hoffmann@inf.ethz.ch>

8.1 Post Office Problem

Suppose there are n post offices $p_1, \ldots p_n$ in a city. Someone who is located at a position q within the city would like to know which post office is closest to him. Modeling the city as a planar region, we think of $p_1, \ldots p_n$ and q as points in the plane. Denote the set of post offices by $P = \{p_1, \ldots p_n\}$. While the locations of post offices are known and do not change so frequently, we do not know in advance for which—possibly many—query locations the closest post office is to be found. Therefore, our long term goal is to come up with a data structure on top of P that allows to answer any possible query efficiently. The basic idea is to apply a so-called *locus approach*: we partition the query space into regions on which is the answer is the same. In our case, this amounts to partition the plane into regions such that for all points within a region the same point from P is closest (among all points from P).

As a warmup, consider the problem for two post offices $p_i, p_j \in P$. For which query locations is the answer p_i rather than p_j? This region is bounded by the bisector of p_i and p_j, that is, the set of points which have the same distance to both points.

Proposition 8.1 *For any two distinct points in $\mathbb{R}^d$ the bisector is a hyperplane, that is, in $\mathbb{R}^2$ it is a line.*

Proof. Let $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ be two points in $\mathbb{R}^d$. The bisector of p and q consists of those points $x = (x_1, \ldots, x_d)$ for which

$$||p - x|| = ||q - x|| \iff ||p - x||^2 = ||q - x||^2 \iff ||p||^2 - ||q||^2 = 2(p - q)x \, .$$

As p and q are distinct, this is the equation of a hyperplane. □

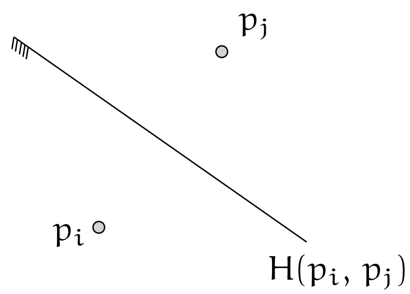


Figure 8.1: *The bisector of two points.*

Denote by $H(p_i, p_j)$ the closed halfplane bounded by the bisector of p_i and p_j that contains p_i.

8.2 Voronoi Diagram

In the following we work with a set $P = \{p_1, \ldots, p_n\}$ of points in $\mathbb{R}^2$.

Definition 8.2 (Voronoi cell) *For* $p_i \in P$ *denote the* **Voronoi cell** $V_P(i)$ *of* p_i *by*

$$V_P(i) := \left\{ q \in \mathbb{R}^2 \,\middle|\, \|q - p_i\| \leq \|q - p\| \text{ for all } p \in P \right\}$$

Proposition 8.3

$$V_P(i) = \bigcap_{j \neq i} H(p_i, p_j) \, .$$

Proof. For $j \neq i$ we have $\|x - p_i\| \leq \|x - p_j\| \iff x \in H(p_i, p_j)$. □

Corollary 8.4 $V_P(i)$ *is non-empty and convex.*

Proof. According to Proposition 8.3 $V_P(i)$ is the intersection of a finite number of halfspaces and hence convex. $V_P(i) \neq \emptyset$ because $p_i \in V_P(i)$. □

Observe that every point of the plane lies in some Voronoi cell but no point lies in the interior of two Voronoi cells. Therefore these cells form a subdivision of the plane.

Definition 8.5 (Voronoi Diagram) *The Voronoi Diagram* $VD(P)$ *of a set* $P = \{p_1, \ldots, p_n\}$ *of points in* $\mathbb{R}^2$ *is the subdivision of the plane induced by the Voronoi cells* $V_P(i)$, *for* $i = 1, \ldots, n$.
Denote by $VV(P)$ *the set of vertices, by* $VE(P)$ *the set of edges, and by* $VR(P)$ *the set of regions (faces) of* $VD(P)$.

To simplify things we restrict our attention to point sets in *general position*, that is,

- no three points from P are collinear and
- no four points from P are cocircular.

Lemma 8.6 *For every vertex* $v \in VV(P)$ *the following statements hold.*

a) v *is the common intersection of exactly three edges from* $VE(P)$;

b) v *is incident to exactly three regions from* $VR(P)$;

c) v *is the center of a circle* $C(v)$ *through exactly three points from* P *such that*

d) $\text{Int}(C(v)) \cap P = \emptyset$.

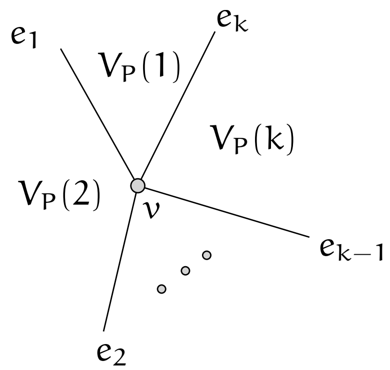


Figure 8.2: *Voronoi regions around* v.

Proof. Consider a vertex $v \in VV(P)$. As all Voronoi cells are convex, $k \geq 3$ of them must be incident to v. Without loss of generality let these cells be $V_P(i)$, for $1 \leq i \leq k$. Denote by e_i, $1 \leq i \leq k$, the edge incident to v that bounds $V_P(i)$ and $V_P((i \bmod k)+1)$. For any $i = 1, \ldots, k$ we have $v \in e_i \Rightarrow |v - p_i| = |v - p_{(i \bmod k)+1}|$. In other words, $p_1, p_2, \ldots, p_k$ are cocircular. Part a), b) and c) follow as a consequence of our general position assumption.

Part d): Suppose there is a point $p_\ell \in \text{Int}(C(v))$. Then v is closer to p_ℓ than to any of $p_1, \ldots, p_k$, in contradiction to the fact that v is incident to all of $V_P(1), \ldots, V_P(k)$. □

Lemma 8.7 *There is an unbounded Voronoi edge bounding* $V_P(i)$ *and* $V_P(j) \iff \overline{p_i p_j}$ *is an edge of* $\text{conv}(P)$.

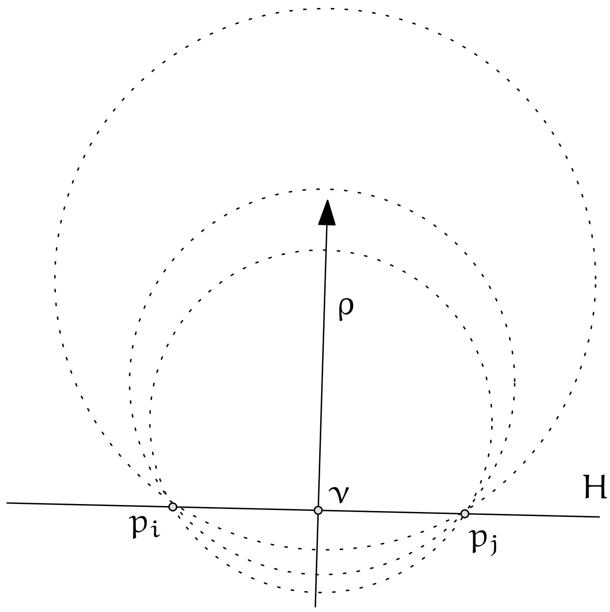
Proof. Consider the family of all circles through p_i and p_j and let

$$\mathcal{C} = \{c_k \mid c_k \text{ is center of the circle through } p_i, p_j, \text{ and } p_k\}.$$

Let H be some closed halfplane through p_i and p_j. As all points from $\mathcal{C}$ lie on the bisector $b_{i,j}$ of p_i and p_j, there is a natural linear order, starting from the point that is furthest away from H up to the point that is furthest inside H. (Clearly there may be no point from $\mathcal{C}$ inside H or no point outside of H, but the order is always well defined.) Let c_k be the maximum from $\mathcal{C}$ according to this order, that is, the point from $\mathcal{C}$ that is located furthest inside H.

The ray ρ starting from c_k along $b_{i,j} \cap H$ is a Voronoi edge. $\iff$ For any $c \in \rho$ there is no point from P closer to c than p_i and p_j. $\iff$ For any $c \in \rho$ the circle centered at c through p_i and p_j does not contain any point from P in its interior. $\iff$ $H \cap P = \{p_i, p_j\}$.

The last statement implies that $\overline{p_i p_j}$ is an edge of $\text{conv}(P)$. For the other direction note that if $\overline{p_i p_j}$ is an edge of $\text{conv}(P)$ then there exists some closed halfplane through p_i and p_j for which $H \cap P = \{p_i, p_j\}$. □



8.3 Duality

Theorem 8.8 (Delaunay 1934) *The straight-line dual of* $\mathrm{VD}(P)$ *for a set* $P \subset \mathbb{R}^2$ *of* $n \geq 3$ *points in general position is a triangulation (→ Delaunay triangulation).*

(Straight-line dual: Graph $G = (P, E)$; $\overline{p_i p_j} \in E \iff |V(i) \cap V(j)| > 1, i \neq j$.)

Proof. Consider $v \in VV(P)$. According to Lemma 8.6b v is incident to exactly three Voronoi regions $V_P(\alpha(v))$, $V_P(\beta(v))$, and $V_P(\gamma(v))$. Let $T(v) := \triangle p_{\alpha(v)} p_{\beta(v)} p_{\gamma(v)}$. Claim:

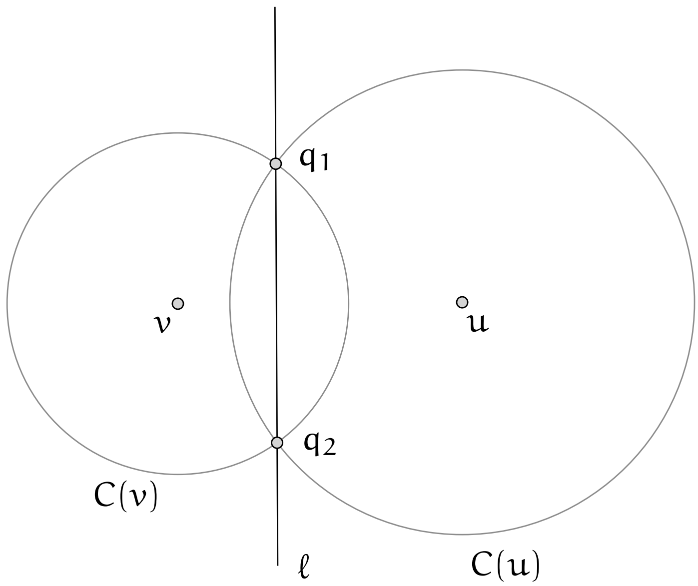
$$\mathcal{T}(P) := \{T(v) \mid v \in VV(P)\}$$

is a triangulation of P.

Obviously $\mathcal{T}(P)$ is a set of triangles whose vertices are from P. The Delaunay property (empty circumcircles) follows from Lemma 8.6d. As $n \geq 3$, we have $\mathcal{T}(P) \neq \emptyset$ by the general position assumption. It remains to show that $\mathcal{T}(P)$ is a triangulation.

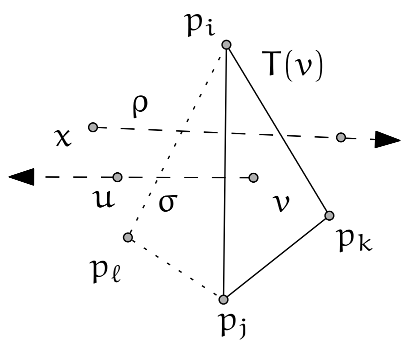
Claim 1. No two triangles from $\mathcal{T}(P)$ have a common interior point.

Suppose two triangles $T(u)$ and $T(v)$, for $u \neq v$, have a common interior point. Denote the circumcircle of $T(u)$ and $T(v)$ by $C(u)$ and $C(v)$, respectively. By Lemma 8.6d neither of $C(u)$ or $C(v)$ is properly contained in the other. Therefore $C(u)$ and $C(v)$ intersect in exactly two points, denote them by q_1 and q_2. By Lemma 8.6d no vertex of $T(u)$ lies in $\mathrm{Int}(C(v))$ and no vertex of $T(v)$ lies in $\mathrm{Int}(C(u))$. But then the line ℓ through q_1 and q_2 separates $T(u)$ and $T(v)$, in contradiction to the assumption that $T(u)$ and $T(v)$ have a common interior point.



Claim 2. Every point of conv(P) lies in some triangle from $\mathcal{T}(P)$.

Consider a point $x \in \text{conv}(P)$ and suppose x is not contained in any triangle from $\mathcal{T}(P)$. Then there is a ray ρ from x that hits an edge $\overline{p_i p_j}$ of some triangle $T(v) = \triangle\, p_i p_j p_k$.



Consider the ray σ starting from v along the bisector of p_i and p_j. An initial segment of σ bounds $V_P(i)$ and $V_P(j)$. As there are points from P on both sides of the line through p_i and p_j, the segment $\overline{p_i p_j}$ is not an edge of conv(P). Therefore, by Lemma 8.7 the boundary between $V_P(i)$ and $V_P(j)$ is not a ray but a line segment $\overline{vu}$. Let $T(u) := \triangle\, p_i p_j p_\ell$. As $\text{Int}(T(u)) \cap \text{Int}(T(v)) = \emptyset$ (s.a.), p_k and p_ℓ lie on different sides of the line through p_i and p_j. But then ρ hits $T(u)$ before $T(v)$, in contradiction to the assumption that $T(v)$ is the first triangle of $\mathcal{T}(P)$ hit by ρ. □

Corollary 8.9 $|VE(P)| \leq 3n - 6$ *and* $|VV(P)| \leq 2n - 5$.

Proof. Every edge in VE(P) corresponds to an edge in the dual Delaunay triangulation. The latter is a plane Graph on n vertices, and thus has at most $3n-6$ edges and at most $2n - 4$ faces. Only the bounded faces correspond to a vertex in VD(P). □

8.4 Lifting Map

Definition 8.10 *Consider the unit paraboloid* $\mathcal{U} : z = x^2 + y^2$ *in* $\mathbb{R}^3$*. Denote by* $u : p = (p_x, p_y, p_z) \mapsto (p_x, p_y, {p_x}^2 + {p_y}^2)$ *the projection of the* x/y*-plane onto* $\mathcal{U}$ *in direction of the positive* z*-axis. This map* $u : \mathbb{R}^2 \to \mathcal{U}$ *is called a lifting map.*

For $p \in \mathbb{R}^2$ let H_p denote the plane of tangency to $\mathcal{U}$ in $u(p)$. Denote by $h_p : \mathbb{R}^3 \to H_p$ the projection of the x/y-plane onto H_p in direction of the positive z-axis.

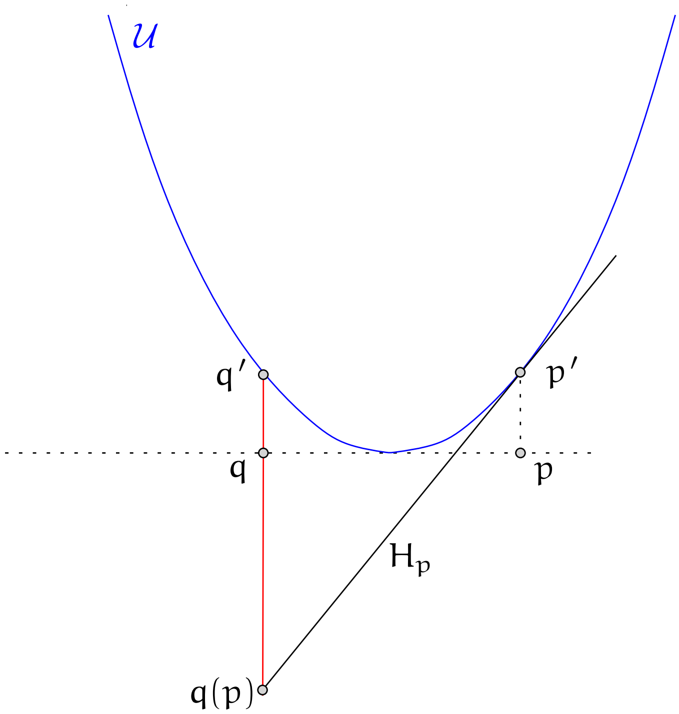


Figure 8.3: *Lifting map* $\mathbb{R}^1 \to \mathbb{R}^2$.

Lemma 8.11 $||u(q) - h_p(q)|| = ||p - q||^2$*, for any points* $p, q \in \mathbb{R}^2$.

Proof. Exercise

Theorem 8.12 *Let* $\mathcal{H}(P) := \bigcap_{p \in P} H_p^+$ *the intersection of all halfspaces above the planes* H_p*,* $p \in P$*. Then the vertical projection of* $\mathcal{H}(P)$ *onto the* x/y*-plane is the Voronoi Diagram of* P*.*

Proof. For any point $q \in \mathbb{R}^2$, the vertical line through q intersects every plane H_p, $p \in P$. By Lemma 8.11 the topmost plane intersected belongs to the point from P that is closest to q. □