9. Point Location

Lecture on Thursday 23rd October, 2008 by Michael Hoffmann <hoffmann@inf.ethz.ch>

Our goal in the following is to provide the missing bit needed to solve the post office problem optimally.

Theorem 9.1 *Given a triangulation* T *for a set* $P \subset \mathbb{R}^2$ *of* n *points, one can build in* $\mathcal{O}(n)$ *time an* $\mathcal{O}(n)$ *size data structure that allows for any query point* $q \in \text{conv}(P)$ *to find in* $\mathcal{O}(\log n)$ *time the triangle from* T *containing* q*.*

Corollary 9.2 (Nearest Neighbor Search) *Given a set* $P \subset \mathbb{R}^2$ *of* n *points, one can build in* $\mathcal{O}(n \log n)$ *time an* $\mathcal{O}(n)$ *size data structure that allows for any query point* $q \in \text{conv}(P)$ *to find in* $\mathcal{O}(\log n)$ *time the nearest neighbor of* q *among the points from* P*.*

9.1 Kirkpatrick's Hierarchy

Idea: Construct a hierarchy $T_0, \ldots, T_h$ of triangulations, such that

- $T_0 = T$,
- the vertices of T_i are a subset of the vertices of T_{i-1}, for $i = 1, \ldots, h$, and
- T_h comprises a single triangle only.

Search. For a query point x the triangle from T containing x can be found as follows.

Search($x \in \mathbb{R}^2$)

1. For $i = h \ldots 0$: Find a triangle t_i from T_i that contains x.
2. return t_h.

This search is efficient under the following conditions.

(C1) Every triangle from T_i intersects only few ($\leq c$) triangles from T_{i-1}. (These will then be connected via the data structure.)

(C2) h is small ($\leq d \log n$).

Proposition 9.3 *The search procedure described above needs* $\leq 3cd \log n = \mathcal{O}(\log n)$ *orientation tests.*

Proof. For every T_i, $0 \leq i < h$, at most c triangles are tested as to whether or not they contain x. □

Thinning. Removing a vertex v and all its incident edges from a triangulation creates a non-triangulated hole that forms a star-shaped polygon since all points are visible from v.

Lemma 9.4 *A starshaped polygon, given as a sequence of* n *vertices and a star-point can be triangulated in* $\mathcal{O}(n)$ *time.*

Proof. Exercise. (The kernel of a simple polygon, that is, the set of all star-points can be constructed in linear time as well.) □

Our working plan is to obtain T_i from T_{i-1} by removing several *independent* (pairwise non-adjacent) vertices and re-triangulating. These vertices should

a) have small degree (otherwise re-triangulating the hole is too expensive) and

b) be many (otherwise the height h of the hierarchy gets too large).

The following lemma asserts the existence of a sufficiently large set of independent small-degree vertices in any triangulation.

Lemma 9.5 *In every triangulation of* n *points in* $\mathbb{R}^2$ *there exists an independent set of at least* $n/18$ *vertices of maximum degree* 8*. Moreover, such a set can be found in* $\mathcal{O}(n)$ *time.*

Proof. Let $T = (V, E)$ denote the graph of the triangulation. We may suppose that T is maximally planar, that is, all faces are triangles. (Otherwise triangulate the exterior face arbitrarily. An independent set in the resulting graph is also independent in T.) For $n = 3$ the statement is true. Let $n \geq 4$.

By the Euler formula we have $|E| = 3n - 6$, that is,

$$\sum_{v \in V} \deg_T(v) = 2|E| = 6n - 12 < 6n.$$

Let $W \subseteq V$ denote the set of vertices of degree at most 8. Claim: $|W| \geq n/2$. Suppose $|W| < n/2$. As every vertex has degree at least three, we have

$$\sum_{v \in V} \deg_T(v) \geq \underbrace{9n/2}_{\text{vertices of degree} \geq 9} + \underbrace{3n/2}_{\text{vertices of degree} \leq 8} \geq 6n,$$

in contradiction to the above.

Construct an independent set U in T as follows (greedily): As long as $W \neq \emptyset$, add an arbitrary vertex $v \in W$ to U and remove v and all its neighbors from W.

Obviously U is independent and all vertices in U have degree at most 8. At each selection step at most 9 vertices are removed from W. Therefore $|U| \geq (n/2)/9 = n/18$. □

Proof. (of Theorem 9.1)

Construct the hierarchy $T_0, \ldots T_h$ with $T_0 = T$ as follows. Obtain T_i from T_{i-1} by removing an independent set U as in Lemma 9.5 and re-triangulating the resulting holes. By Lemma 9.4 and Lemma 9.5 every step is linear in the number $|T_i|$ of vertices in T_i. The total cost for building the data structure is thus

$$\sum_{i=0}^{h} \alpha |T_i| \leq \sum_{i=0}^{h} \alpha n (17/18)^i < 18 \alpha n = \mathcal{O}(n),$$

for some constant α. Similarly the space consumption is linear.

The number of levels amounts to $h = \log_{18/17} n < 12.2 \log n$. Thus by Proposition 9.3 the search needs at most $3 \cdot 8 \cdot \log_{18/17} n < 292 \log n$ orientation tests. □

Improvements. As the name suggests, the hierarchical approach discussed above is due to David Kirkpatrick [?]. The constant 292 that appears in the search time is somewhat large. There has been a whole line of research trying to improve it using different techniques.

- Sarnak and Tarjan [?]: $4 \log n$.
- Edelsbrunner, Guibas, and Stolfi [?]: $3 \log n$.
- Goodrich, Orletsky, and Ramaier [?]: $2 \log n$.
- Adamy and Seidel [?]: $1 \log n + 2\sqrt{\log n}$.

References