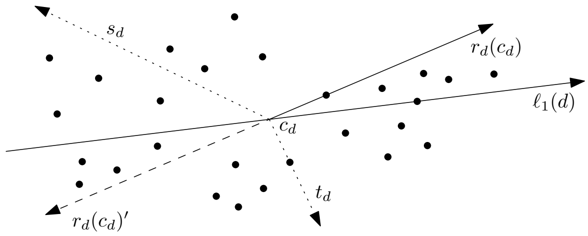
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Prof. Dr. B. Gärtner, A. Gundert,
Dr. M. Hoffmann, Prof. Dr. E. Welzl
URL: http://www.ti.inf.ethz.ch/ew/courses/CG13/

Computational Geometry — Homework 3 — HS13

Exercise 1 (30 Points)

Let P be a set of n points in the plane in general position. The goal of this exercise is to prove the following statement, which is the discrete variant of a result by R.C. and Ellen F. Buck (1949): There are 3 lines ℓ_1, ℓ_2 and ℓ_3 passing through a common point c dividing the plane into 6 sectors such that each sector contains no more than $n/6$ points in its interior.



Hint: Follow the argumentation sketched below. Reasoning about basic geometric ideas can be tricky. Since the basic idea is already given, be careful to describe your solution exactly and in sufficient detail. Your goal should be to convince the reader that you really understand why this works!

For a ray or a directed line a, let a' denote the ray or line pointing into the opposite direction.

(a) Show that for every possible direction there is a line bisecting P. More precisely, show that there is a continuous function mapping each direction $d \in [0, 2\pi]$ to a directed line $\ell_1(d)$ with direction d that bisects P. This means neither the open halfplane to the left of $\ell_1(d)$ nor the open halfplane to the right contains more than $n/2$ points.

(b) Show that for each point q on $\ell_1(d)$ there is a ray $r_d(q)$ starting at q such that the sector between $\ell_1(d)$ and $r_d(q)$ (defined counter-clockwise) contains at most $1/6$ of the points and the sector between $r_d(q)$ and $\ell_1(d)'$ contains at most $1/3$ of the points. Argue that there is a point c_d on $\ell(d)$ such that at the same time the sector between $\ell(d)'$ and $r_d(c_d)'$ contains at most $1/6$ of the points and the sector between $r_d(c_d)'$ and $\ell_1(d)$ at most $1/3$.

(c) $r_d(c_d)$ can be constructed in such a way that the directed line $\ell_2(d)$ defined by $r_d(c_d)$ depends continuously on d.

(d) Let s_d be a ray starting at c_d bisecting the sector between $\ell_2(d)$ and $\ell_1(d)'$ and let t_d be a ray starting at c_d bisecting the opposite sector between $\ell_2(d)'$ and $\ell_1(d)$. (By bisecting we mean that there at most $1/6$ of the points in the open sector between $\ell_1(d)'$ and s_d and at most $1/6$ of the points in the open sector between s_d and $\ell_2(d)$.) Show that both s_d and t_d can be constructed to be continuous in d.

(e) Conclude that there must be a direction d^* such that s_{d^*} and t_{d^*} lie on a common line ℓ_3.

Exercise 2 (10 Points)

Describe an $O(n^2)$ time algorithm that given a set P of n points in the plane finds a subset of five points that form a strictly convex empty pentagon (or reports that there is none if that is the case). Empty means that the convex pentagon may not contain any other points of P.

Hint: For each $p \in P$ discard all points left of p and consider the polygon $S(p)$ formed by p and the remaining points taken in circular order around p. Explain why it suffices to check for all p whether $S(p)$ has 4 vertices other than p that form an empty convex quadrilateral. How do you check this in $O(n^2)$ time?

Due date: 12.12.2013, 13h15