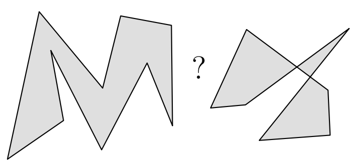
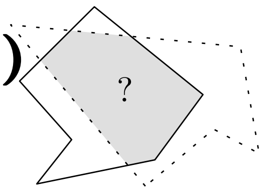
Problem 1
Does $P = (p_1, \ldots, p_n)$ *form a simple Polygon?*

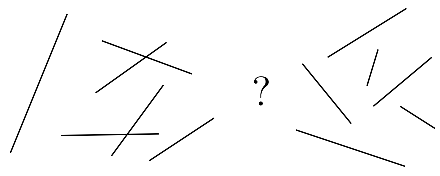


Problem 2 (Polygon Intersection)
Do two simple polygons intersect?



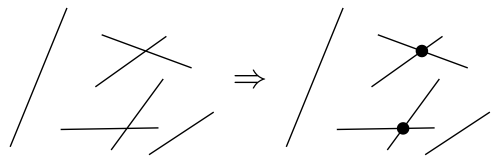
Problem 3 (Segment Intersection Test)

Are n *line segments pairwise disjoint?*



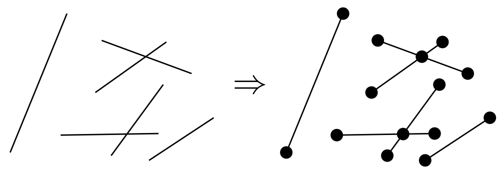
Problem 4 (Segment Intersection)

Given n line segments, construct all intersections.



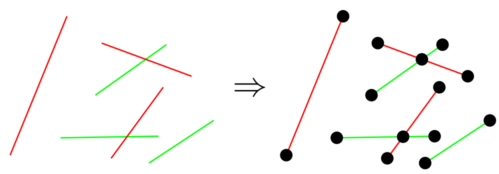
Problem 5 (Segment Arrangement)

Given n line segments, construct their Arrangement.



Problem 6 (Map Overlay)

Given sets S and T of pairwise disjoint line segments, construct the Arrangement of $S \cup T$.



Segment Intersection

Trivial Algorithm. Test all pairs. $O(n^2)$ time and linear space.
Worst-case optimal for Problem 4...

In case of few intersections, we would like to have sub-quadratic time.

Lower bound $\Omega(n \log n)$ from Element Uniqueness.

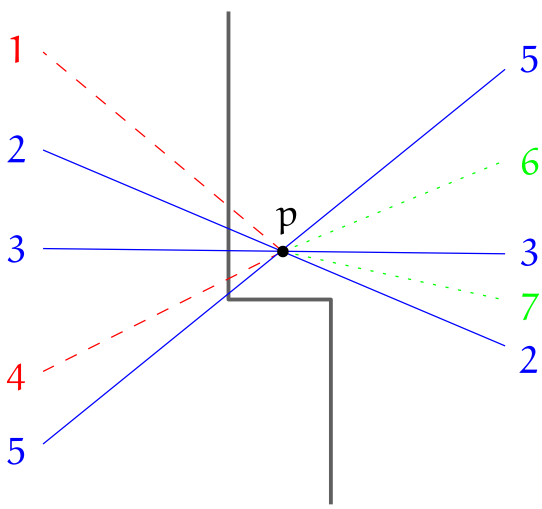
Problem 7 Given a set I of n intervals $[\ell_i, r_i] \subset \mathbb{R}$, $1 \leq i \leq n$, compute all intersecting pairs.

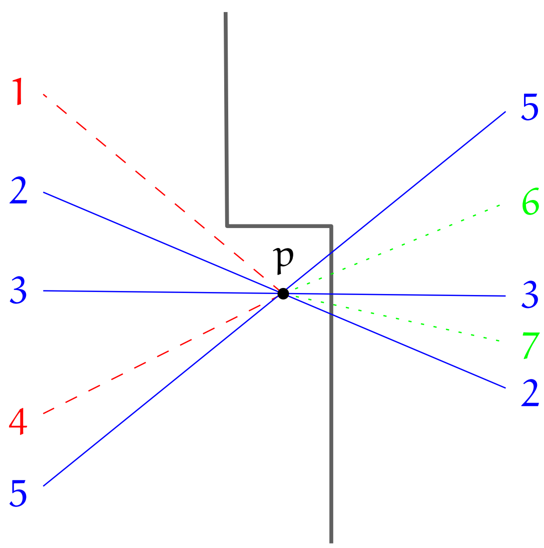
Theorem 1
Problem 7 can be solved in $O(n \log n + k)$ time and $O(n)$ space, where k is the number of intersecting pairs.

Line Sweep

Idea. Move a line ℓ (*sweep line*) from left to right, such that at any time all intersections to the left of ℓ are known.

We do not make any general position assumption here, that is, several segments can start, end, and/or intersect at the same point. Imagine the sweep line infinitesimally twisted.





Sweep line status (SLS). Sequence L of segments that intersect the current sweep line, sorted by y-coordinate.

Event point (EP). Point where SLS changes when moving ℓ (discretization).

Event point schedule (EPS). Sequence E of event points to be processed (not all known in advance), sorted lexicographically.

With every EP p we store

- a list end(p) of segments ending at p;
- a list begin(p) of segments that begin at p;
- a list int(p) of segments that intersect a neighboring (in SLS) segment at p.

With every segment we store pointers to the (≤ 2) entries in int$(\cdot)$ lists and a pointer to its appearance in L.

Invariants.

i) L is the sequence of segments from S which intersect ℓ, sorted by y-coordinate ($\leq$);

ii) E contains all event points (endpoints from segments in S and all points of intersection from segments adjacent in L) that are to the right of ℓ;

iii) All intersections between segments from S that are to the left of ℓ have been reported.

Event point handling. Consider an EP p.

1) If end$(p) \cup$ int$(p) = \emptyset$, localize p in L.

2) Report all pairs of segments from end$(p)\cup$ begin$(p) \cup$ int(p) as intersecting.

3) Remove all segments in end(p) from L.

4) Reverse the subsequence in L that is formed by the segments from int(p).

5) Insert segments from begin(p) into L, sorted by slope.

6) Test the topmost and bottommost segment in SLS from begin$(p) \cup$ int(p) for intersection with its successor and predecessor, respectively, and update EP if necessary.

Update of EPS. Insert an EP p for intersection of segments s and t.

1) If p does not yet appear in E, insert it.

2) If s or t are contained in some int$(\cdot)$ list of some other EP q, remove them there and possibly remove q from E (if end$(q) \cup$ begin$(q) \cup$ int$(q) = \emptyset$).

3) Insert s and t into int(p).

Sweep.

1) Insert all segment endpoints into begin$(\cdot)$ and end$(\cdot)$ lists of a corresponding EP in E.

2) As long as $E \neq \emptyset$, handle the first EP and then remove it from E.

Runtime Analysis

Initialization: $O(n \log n)$.

Handling of an EP p:

$$O(\#\text{intersecting pairs} + |\text{end}(p)| \log n + |\text{int}(p)| + |\text{begin}(p)| \log n + \log n).$$

Altogether:

$$O(k + n \log n + k \log n) = O((n + k) \log n).$$

Space. Clearly $|S| \leq n$. At begin $|E| \leq 2n$ and $|S| = 0$. Never more than $2|S|$ intersection EPs, therefore linear space overall.

Theorem 2 Problem 4 and Problem 5 can be solved in $O((n + k) \log n)$ time and $O(n)$ space.

Theorem 3 Problem 1, Problem 2 and Problem 3 can be solved in $O(n \log n)$ time and $O(n)$ space.

Improvements

The presented algorithm is due to Jon Bentley and Thomas Ottmann (1979).

$O(n \log n + k)$ time and $O(n+k)$ space [Bernard Chazelle and Herbert Edelsbrunner (1988)]

expected $O(n \log n + k)$ time using $O(n + k)$ space [Ketan Mulmuley (1988)]

expected $O(n \log n + k)$ time using $O(n)$ space [Kenneth Clarkson and Peter Shor (1989)]

$O(n \log n + k)$ time and linear space [Ivan Balaban (1995)]