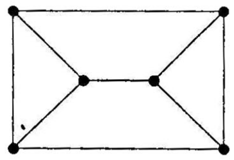
Graph Theory

Homework exercises # 1. — Due Oct 30.

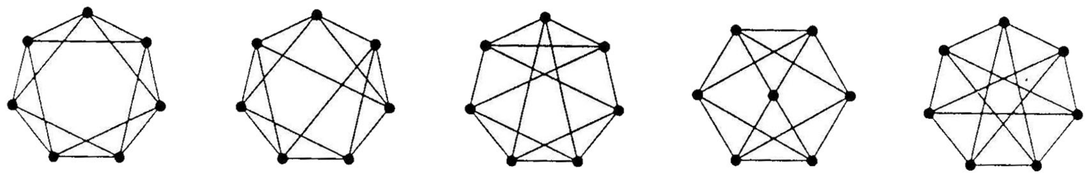
1.1.6. (−) Determine whether the graph below decomposes into copies of P_4.



1.1.7. (−) Prove that a graph with more than six vertices of odd degree cannot be decomposed into three paths.

1.1.14. (!) Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.

1.1.22. (!) Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.



1.1.27. (!) Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with exactly $k^2 + 1$ vertices.

1.1.29. Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers.

1.1.31. (!) Prove that a self-complementary graph with n vertices exists if and only if n or $n - 1$ is divisible by 4. (Hint: When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1 \bmod 4$, add one vertex to the graph constructed for $n - 1$.)

1.1.47. (∗) *Edge-transitive versus vertex-transitive.*

a) Let G be obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2. Prove that G is edge-transitive but not vertex-transitive.

b) Suppose that G is edge-transitive but not vertex-transitive and has no vertices of degree 0. Prove that G is bipartite.

c) Prove that the graph in Exercise 1.1.6 is vertex-transitive but not edge-transitive.

1.1.28. (+) *The Odd Graph O_k.* The vertices of the graph O_k are the k-element subsets of $\{1, 2, \ldots, 2k + 1\}$. Two vertices are adjacent if and only if they are disjoint sets. Thus O_2 is the Petersen graph. Prove that the girth of O_k is 6 if $k \geq 3$.