Complete k-partite graphs

G is a complete k-partite graph if there is a partition $V_1 \cup \dots V_k = V(G)$ of the vertex set, such that $uv \in E(G)$ iff u and v are in *different* parts of the partition. If $|V_i| = n_i$, then G is denoted by $K_{n_1,\dots,n_k}$.

The Turán graph $T_{n,r}$ is the complete r-partite graph on n vertices whose partite sets differ in size by at most 1. (All partite sets have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.)

Lemma Among r-colorable graphs the Turán graph is the *unique* graph, which has the most number of edges.

Proof. Local change.

Turán's Theorem

The Turán number $ex(n, H)$ of a graph H is the largest integer m such that there exists an H-free* graph on n vertices with m edges.

Example: Mantel's Theorem states $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Theorem. (Turán, 1941)

$$ex(n, K_r) = e(T_{n,r-1}) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + O(n).$$

Proof. Prove by induction on r that

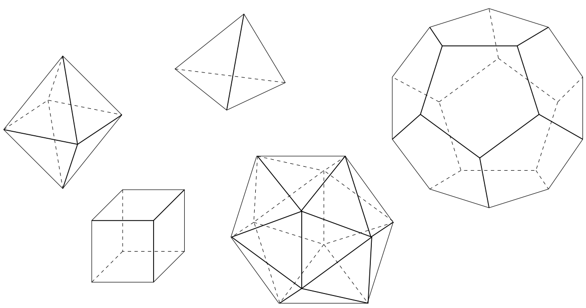
$G \not\supseteq K_r \implies$ there is an $(r-1)$-partite graph H with $V(H) = V(G)$ and $e(H) \geq e(G)$.

Then apply the Lemma to finish the proof.

*Here H-free means that there is no subgraph isomorphic to H

Turán-type problems

Question. (Turán, 1941) What happens if instead of K_4, which is the graph of the tetrahedron, we forbid the graph of some other platonic polyhedra? How many edges can a graph without an octahedron (or cube, or dodecahedron or icosahedron) have?



The platonic solids

Erdős-Simonovits-Stone Theorem

Theorem. (Erdős, Simonovits, Stone) For any graph H,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Corollaries.

$$\begin{aligned}
ex(n, \texttt{octahedron}) &= \frac{n^2}{4} + o(n^2) \\
ex(n, \texttt{dodecahedron}) &= \frac{n^2}{4} + o(n^2) \\
ex(n, \texttt{icosahedron}) &= \frac{n^2}{3} + o(n^2) \\
ex(n, \texttt{cube}) &= o(n^2)
\end{aligned}$$

Open problems and Conjectures

Known results.

$$\begin{aligned}\Omega(n^{3/2}) &\le ex(n, Q_3) &\le O(n^{8/5})\\ \Omega(n^{9/8}) &\le ex(n, C_8) &\le O(n^{5/4})\\ \Omega(n^{5/3}) &\le ex(n, K_{4,4}) &\le O(n^{7/4})\end{aligned}$$

Conjectures.

$$ex(n, K_{t,s}) = \Theta\left(n^{2-\frac{1}{\min\{t,s\}}}\right) \text{ true for } s = 2,3 \text{ and } t \ge s \text{ or } s \ge 4 \text{ and } t > (s-1)!$$

$$ex(n, C_{2k}) = \Theta\left(n^{1+\frac{1}{k}}\right) \text{ true for } k = 2,3 \text{ and } 5.$$

$$ex(n, Q_3) = \Theta\left(n^{\frac{8}{5}}\right)$$

If H is a d-degenerate bipartite graph, then

$$ex(n, H) = O\left(n^{2-\frac{1}{d}}\right).$$

Proof of the ESS Theorem

Theorem. (Erdős-Stone, 1946) For arbitrary fixed integers $r \ge 2$ and $t \ge 1$

$$ex(n, T_{rt,r}) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2).$$

Proof of ESS from Erdős-Stone Theorem.
(Erdős-Simonovits, 1966)
Let $r = \chi(H)$.

- $\chi(T_{n,r-1}) < \chi(H)$, so $e(T_{n,r-1}) \le ex(n, H)$.
- $T_{r\alpha,r} \supseteq H$, so $ex(n, T_{r\alpha,r}) \ge ex(n, H)$, where α is a constant depending on H; say $\alpha = \alpha(H)$.

Proof of the Erdős-Stone Theorem

Erdős-Stone Theorem. (Reformulation) For any $\epsilon > 0$ and integers $r \ge 2$, $t \ge 1$ there exists an integer $M = M(r, t, \epsilon)$, such that any graph G on $n \ge M$ vertices with more than $\left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2}$ edges contains $T_{rt,r}$.

We derive this through the following statement.

Seemingly Weaker Theorem. For any $\epsilon > 0$ and integers $r \ge 2$, $t \ge 1$ there exists an integer $N = N(r, t, \epsilon)$, such that any graph G on $n \ge N$ vertices and with $\delta(G) \ge \left(1 - \frac{1}{r-1} + \epsilon\right) n$ contains $T_{rt,r}$.

Note that w.l.o.g. $\epsilon < \frac{1}{r-1}$.

Derivation of the Erdős-Stone Theorem from the Seemingly Weaker Theorem.

Let G be a graph on $n \ge M(r, t, \epsilon)^*$ vertices with more than $\left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2}$ edges. Recursively delete vertices which are adjacent to less than $\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)$-fraction of the remaining vertices.
What is the number n' of vertices we are left with?

We deleted at most $\sum_{j=n'+1}^{n} j\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right)$ edges. So

$$e(G) \le \left(\binom{n+1}{2} - \binom{n'+1}{2}\right)\left(1 - \frac{1}{r-1} + \frac{\epsilon}{2}\right) + \binom{n'}{2}.$$

This implies

$$\frac{\epsilon}{2}\binom{n}{2} - n \le \left(\frac{1}{r-1} - \frac{\epsilon}{2}\right)\binom{n'}{2} - n'.$$

We choose $M(r, t, \epsilon)$ such that $n \ge M(r, t, \epsilon)$ implies $n' \ge N(r, t, \epsilon/2)$.

*At this point we don't know $M(r, t, \epsilon)$ yet!!! We'll define it in the proof through $N(r, t, \epsilon/2)$. (which is known!)

Proof of the Seemingly Weaker Theorem.
Induction on r.
For $r = 2$ the claim is true provided $\frac{\binom{\epsilon n}{t} n}{\binom{n}{t}} > t - 1$, which is certainly true from some threshold $N(2, t, \epsilon)$.

Let $r \geq 2$ and G be a graph on $n \geq N(r+1, t, \epsilon)^*$ vertices with $\delta(G) \geq \left(1 - \frac{1}{r} + \epsilon\right) n$.
We would like to find a $T_{(r+1)t,r+1}$ in G.

Let $s = \left\lceil \frac{t}{\epsilon} \right\rceil$. By the induction hypothesis† there is a $T_{rs,r}$ in G with vertex-set $A_1 \cup \ldots \cup A_r$, where $|A_1| = \ldots = |A_r| = s$.

$U = V(G) \setminus (A_1 \cup \ldots \cup A_r)$.

$W = \{w \in U : |N(w) \cap A_i| \geq t, i = 1, \ldots, r\}$ is the set of vertices eligible to extend some part of $A_1, \ldots, A_r$ into a $T_{(r+1)t,r+1}$.

*Again, we don't know $N(r+1, t, \epsilon)$ yet.
†Here we assume $N(r+1, t, \epsilon) \geq N(r, s, \epsilon)$.

Double-count the number of edges missing between U and $A_1 \cup \ldots \cup A_r$. They are

- at least $(|U| - |W|)(s - t)$ and
- at most $rs\left(\frac{1}{r} - \epsilon\right) n$.

From this we have

$$|W| \geq \frac{(r-1)\epsilon}{1-\epsilon} n - rs$$

Thus if n is large enough* then

$$|W| > \binom{s}{t}^r (t-1).$$

So we can select t vertices from W, which are adjacent to the same t vertices in each A_i .

*If $N(r+1, t, \epsilon) > \left(\binom{s}{t}^r (t-1) + rs\right) \frac{1-\epsilon}{(r-1)\epsilon}$