Matchings

A matching is a set of (non-loop) edges with no shared endpoints. The vertices incident to an edge of a matching M are saturated by M, the others are unsaturated. A perfect matching of G is matching which saturates all the vertices.

Examples. $K_{n,m}, K_n$, Petersen graph, Q_k; graphs without perfect matching

A maximal matching cannot be enlarged by adding another edge.

A maximum matching of G is one of maximum size.

Example. Maximum $\neq$ Maximal

Characterization of maximum matchings

Let M be a matching. A path that alternates between edges in M and edges not in M is called an M-alternating path.
An M-alternating path whose endpoints are unsaturated by M is called an M-augmenting path.

Theorem(Berge, 1957) A matching M is a maximum matching of graph G iff G has no M-augmenting path.

Proof. ($\Rightarrow$) Easy.
($\Leftarrow$) Suppose there is no M-augmenting path and let M^* be a matching of maximum size.
What is then $M \triangle M^*$???

Lemma Let M_1 and M_2 be matchings of G. Then each connected component of $M_1 \triangle M_2$ is a path or an even cycle.

For two sets A and B, the symmetric difference is $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Hall's Condition and consequences

Theorem (Marriage Theorem; Hall, 1935) Let G be a bipartite (multi)graph with partite sets X and Y. Then there is a matching in G saturating X iff $|N(S)| \geq |S|$ for every $S \subseteq X$.

Proof. ($\Rightarrow$) Easy.

($\Leftarrow$) Not *so* easy. Find an M-augmenting path for *any* matching M which does not saturate X.
(Let U be the M-unsaturated vertices in X. Define

$$T := \{y \in Y : \exists\ M\text{-alternating } U, y\text{-path}\},$$
$$S := \{x \in X : \exists\ M\text{-alternating } U, x\text{-path}\}.$$

Unless there is an M-augmenting path, $S \cup U$ **violates** Hall's condition.)

Corollary. (Frobenius (1917)) For $k > 0$, every k-regular bipartite (multi)graph has a perfect matching.

Graph parameters — Definitions and simple properties

The size of the largest independent set in G is denoted by $\alpha(G)$.

The size of the largest matching (*independent set of edges*) in G is denoted by $\alpha'(G)$.

A vertex cover of G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. (The vertices in Q *cover* $E(G)$).
The size of the smallest vertex cover in G is denoted by $\beta(G)$.

Claim. $\beta(G) \geq \alpha'(G)$.

An edge cover of G is a set L of edges such that every vertex of G is incident to some edge in L.
The size of the smallest edge cover in G is denoted by $\beta'(G)$.

Claim. $\beta'(G) \geq \alpha(G)$.

Certificates

Suppose we knew that in some graph G with 1121 edges on 200 vertices, a particular set of 87 edges is (one of) the largest matching one could find. How could we convince somebody about this?

Once the particluar 87 edges are shown, it is easy to check that they are a matching, indeed.

But why isn't there a matching of size 88? Verifying that none of the $\binom{1121}{88}$ edgesets of size 88 forms a matching could take some time...

If we happen to be so lucky, that we are able to exhibit a vertex cover of size 87, we are saved. It is then reasonable to check that all 1121 edges are covered by the particular set of 87 vertices.

Exhibiting a vertex cover of a certain size **proves** that no larger matching can be found.

Min-max theorems for bipartite graphs

Theorem. (König (1931), Egerváry (1931)) If G is bipartite then $\beta(G) = \alpha'(G)$.

Proof. For any minimum vertex cover Q, apply Hall's Condition to match $Q \cap X$ into $Y \setminus Q$ and $Q \cap Y$ into $X \setminus Q$.

Lemma. Let G be any graph. $S \subseteq V(G)$ is an independent set iff $\overline{S}$ is a vertex cover.
Hence $\alpha(G) + \beta(G) = n(G)$.

Proof. Easy.

Theorem. (Gallai, 1959) Let G be any graph without isolated vertices. Then $\alpha'(G) + \beta'(G) = n(G)$.

Corollary. (König, 1916) Let G be a bipartite graph with no isolated vertices. Then $\alpha(G) = \beta'(G)$.

Proof. Put together the previous three statements.

How to find a maximum matching in bipartite graphs?

Augmenting Path Algorithm

Input. A bipartite graph G with partite sets X and Y,
a matching M in G,
the set U of unsaturated vertices in X.

Output. EITHER an M-augmenting path OR a certificate (a cover of the same size) that M is maximum.

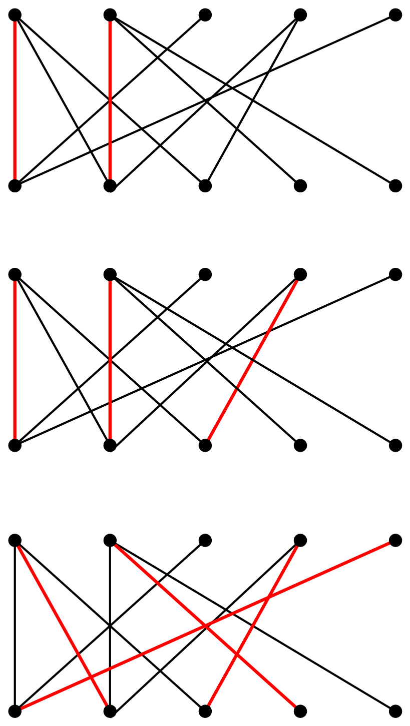
Idea. Explore M-alternating paths from U, letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached. Mark vertices of S that have been explored for path extensions. As a vertex is reached, record the vertex from which it is reached.

Initialization. $S = U$ and $T = \emptyset$.

Iteration.
IF all vertices in S are marked THEN
    stop and **report** that M is a maximum matching
    and $T \cup (X \setminus S)$, is a cover of the same size.
ELSE
    select an unmarked $x \in S$ and explore its neighbors $y \in N(x)$, for which $xy \notin M$.
    IF y is unsaturated, THEN
        stop and **report** an M-augmenting path from U to y.
    ELSE
        $\exists w \in X$ with $yw \in M$. **Update**
            $T := T \cup \{y\}$ (y is reached from x),
            $S := S \cup \{w\}$ (w is reached from y).
    After exploring all neighbors of x, mark x and
iterate.

Theorem. Repeatadly applying the Augmenting Path Algorithm to a bipartite graph produces a maximum matching and a minimum vertex cover.

If G has n vertices and m edges, then this algorithm finds a maximum matching in $O(nm)$ time.



Proof of correctness

If Augmenting Path Algorithm does what it supposed to, then after at most $n/2$ application we can produce a maximum matching.
Why does the APA terminate? It touches each edge at most once. Hence running time is $O(nm)$.

What if an M-augmenting path is returned? It is OK, since y is an unsaturated neighbor of $x \in S$, and x can be reached from U on an M-alternating path.

What if the APA returns M as maximum matching and $T \cup (X \setminus S)$ as minimum cover?

Then all edges leaving S were explored, so there is **no edge between S and $Y \setminus T$**.

- Hence $T \cup (X \setminus S)$ is indeed a cover.
- $|M| = |T| + |X \setminus S|$ (By selection of S and T.)

If a cover and a matching have the same size in any graph, then they are both optimal.

$$|M| \le \alpha'(G) \le \beta(G) \le |T \cup (X \setminus S)| = |M|.$$