## How to find a maximum weight matching in a bipartite graph?

In the maximum weighted matching problem a non-negative weight $w_{i,j}$ is assigned to each edge $x_i y_j$ of $K_{n,n}$ and we seek a perfect matching $M$ to maximize the total weight $w(M) = \sum_{e \in M} w(e)$.

With these weights, a (weighted) cover is a choice of labels $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, such that $u_i + v_j \geq w_{i,j}$ for all $i, j$. The cost $c(u, v)$ of a cover $(u, v)$ is $\sum u_i + \sum v_j$. The minimum weighted cover problem is that of finding a cover of minimum cost.

**Duality Lemma** For a perfect matching $M$ and a weighted cover $(u, v)$ in a bipartite graph $G$, $c(u, v) \geq w(M)$. Also, $c(u, v) = w(M)$ iff $M$ consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$. In this case, $M$ and $(u, v)$ are both optimal.

# The algorithm

The equality subgraph $G_{u,v}$ for a weighted cover $(u, v)$ is the spanning subgraph of $K_{n,n}$ whose edges are the pairs $x_i y_j$ such that $u_i + v_j = w_{i,j}$. In the cover, the excess for $i, j$ is $u_i + v_j - w_{i,j}$.

## Hungarian Algorithm

**Input.** A matrix $(w_{i,j})$ of weights on the edges of $K_{n.n}$ with partite sets $X$ and $Y$.

**Idea.** Iteratively adjusting a cover $(u, v)$ until the equality subgraph $G_{u,v}$ has a perfect matching.

**Initialization.** Let $u_i = \max\{w_{i,j} : j = 1, \ldots, n\}$ and $v_j = 0$.

**Iteration.**

Form $G_{u,v}$ and find a maximum matching $M$ in it.

IF $M$ is a perfect matching, THEN

**stop** and **report** $M$ as a maximum weight matching and $(u, v)$ as a minimum cost cover

ELSE

let $Q$ be a vertex cover of size $|M|$ in $G_{u,v}$.

$R := X \cap Q$

$T := Y \cap Q$

$\epsilon :=$ min$\{u_i + v_j - w_{i,j} : x_i \in X \setminus R, y_j \in Y \setminus T\}$

**Update** $u$ and $v$:

$u_i := u_i - \epsilon$ if $x_i \in X \setminus R$

$v_j := v_j + \epsilon$ if $y_j \in T$

**Iterate**

**Theorem** The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover.

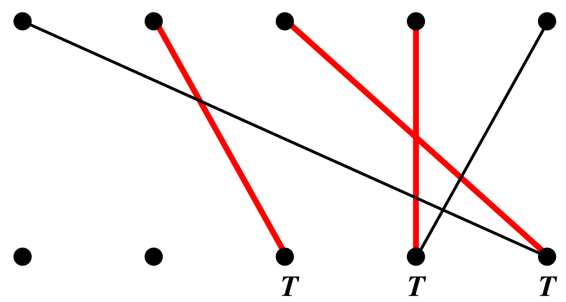
# The Assignment Problem — An example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 7 & 2 \\ 1 & 3 & 4 & 4 & 5 \\ 3 & 6 & 2 & 8 & 7 \\ 4 & 1 & 3 & 5 & 4 \end{pmatrix}$$

Excess Matrix

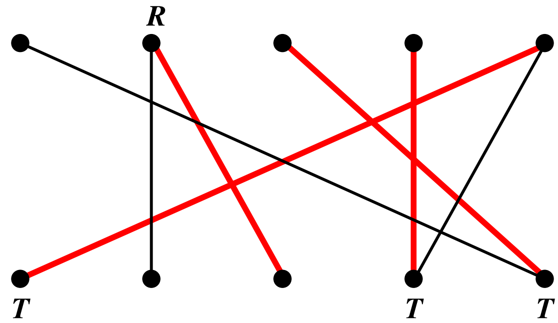
$$\begin{matrix} & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 8 & 2 & 1 & 0 & 1 & 6 \\ 5 & 4 & 2 & 1 & 1 & 0 \\ 8 & 5 & 2 & 6 & 0 & 1 \\ 5 & 1 & 4 & 2 & 0 & 1 \\ & & & T & T & T \end{matrix}$$

Equality Subgraph



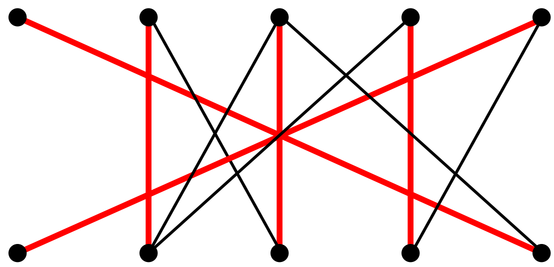

$\epsilon = 1$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \end{array} \\
\begin{array}{c} 4 \\ 7 \\ 4 \\ 7 \\ 4 \end{array} &
\left( \begin{array}{ccccc}
3 & \color{blue}{2} & \color{blue}{2} & 1 & 0 \\
1 & 0 & 0 & 1 & 6 \\
3 & \color{green}{1} & \color{green}{1} & 1 & 0 \\
4 & \color{green}{1} & \color{blue}{6} & 0 & 1 \\
0 & \color{blue}{3} & \color{blue}{2} & 0 & 1
\end{array} \right)
\begin{array}{c} \\ R \\ \\ \\ \\ \end{array} \\
 & \begin{array}{ccccc} T & & & T & T \end{array}
\end{array}
$$




$$\color{green}{\epsilon = 1}$$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 0 & 1 & 2 & 2 \end{array} \\
\begin{array}{c} 3 \\ 7 \\ 3 \\ 6 \\ 3 \end{array} &
\left( \begin{array}{ccccc}
3 & 1 & 1 & 1 & 0 \\
2 & 0 & 0 & 2 & 7 \\
3 & 0 & 0 & 1 & 0 \\
4 & 0 & 5 & 0 & 1 \\
0 & 2 & 1 & 0 & 1
\end{array} \right)
\end{array}
$$



DONE!!

The Duality Lemma states that if $w(M) = c(u,v)$ for some cover $(u,v)$, then $M$ is maximum weight.

We found a maximum weight matching (transversal). The fact that it is maximum is certified by the indicated cover, which has the same cost:

$$\begin{array}{c} \\ 3 \\ 7 \\ 3 \\ 6 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 0 & 1 & 2 & 2 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} \\ 6 & \mathbf{7} & 8 & 7 & 2 \\ 1 & 3 & \mathbf{4} & 4 & 5 \\ 3 & 6 & 2 & \mathbf{8} & 7 \\ \mathbf{4} & 1 & 3 & 5 & 4 \end{pmatrix} \end{array}$$

$$\begin{aligned} w(M) &= 5+7+4+8+4 = 28 = \\ &= 1+0+1+2+2+ \\ &\quad 3+7+3+6+3 = c(u,v) \end{aligned}$$

# Hungarian Algorithm — Proof of correctness

*Proof.* If the algorithm ever terminates and $G_{u,v}$ is the equality subgraph of a $(u, v)$, which is indeed a cover, then $M$ is a m.w.m. and $(u, v)$ is a m.c.c. by Duality Lemma.

*Why is $(u, v)$, created by the iteration, a cover?*

Let $x_i y_j \in E(K_{n,n})$. Check the four cases.

| | | | |
|---|---|---|---|
| $x_i \in R,$ | $y_j \in Y \setminus T$ | $\Rightarrow$ | $u_i$ and $v_j$ do not change. |
| $x_i \in R,$ | $y_j \in T$ | $\Rightarrow$ | $u_i$ does not change<br>$v_j$ increases. |
| $x_i \in X \setminus R,$ | $y_j \in T$ | $\Rightarrow$ | $u_i$ decreases by $\epsilon$,<br>$v_j$ increases by $\epsilon$. |
| $x_i \in X \setminus R,$ | $y_j \in Y \setminus T$ | $\Rightarrow$ | $u_i + v_j \geq w_{i,j}$<br>by definition of $\epsilon$. |

*Why does the algorithm terminate?*

$M$ is a matching in the new $G_{u,v}$ as well. So either

$(i)$ max matching gets larger or

$(ii)$ $\#$ of vertices reached from $U$ by $M$-alternating paths grows. ($U$ is the set of unsaturated vertices of $M$ in $X$.)

## Matchings in general graphs

An odd component is a connected component with an odd number of vertices. Denote by $o(G)$ the number of odd components of a graph $G$.

**Theorem.** (Tutte, 1947) A graph $G$ has a perfect matching iff $o(G - S) \leq |S|$ for every subset $S \subseteq V(G)$.

*Proof.*

$\Rightarrow$ Easy.

$\Leftarrow$ (Lovász, 1975) Consider a counterexample $G$ with the maximum number of edges.

*Claim.* $G + xy$ has a perfect matching for any $xy \notin E(G)$.

# Proof of Tutte's Theorem — Continued

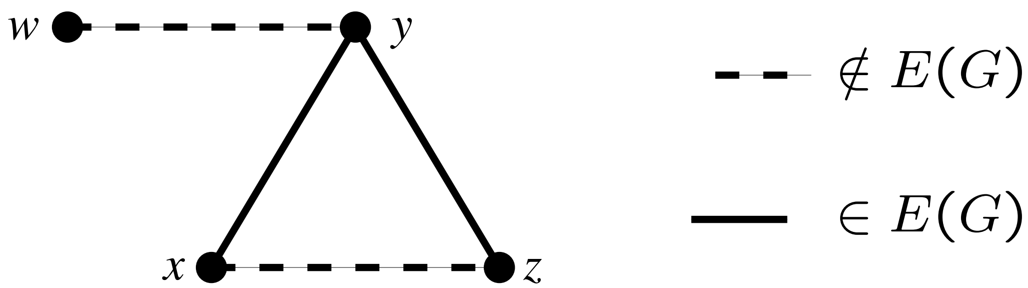
Define $U := \{v \in V(G) : d_G(v) = n(G) - 1\}$

*Case 1.* $G - U$ consists of disjoint cliques.

*Proof:* Straightforward to construct a perfect matching of $G$.

*Case 2.* $G - U$ is not the disjoint union of cliques.

*Proof:* Derive the existence of the following subgraph.




Obtain contradiction by constructing a perfect matching $M$ of $G$ using perfect matchings $M_1$ and $M_2$ of $G + xz$ and $G + yw$, respectively.

## Corollaries

**Corollary.** (Berge,1958) For a subset $S \subseteq V(G)$ let $d(S) = o(G - S) - |S|$. Then

$$2\alpha'(G) = \min\{n - d(S) : S \subseteq V(G)\}.$$

*Proof.* ($\leq$) Easy.
($\geq$) Apply Tutte's Theorem to $G \vee K_d$.

**Corollary.** (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

*Proof.* Check Tutte's condition. Let $S \subseteq V(G)$.
Double-count the number of edges between an $S$ and the odd components of $G - S$.
Observe that between any odd component and $S$ there are at least three edges.

## Factors

A factor of a graph is a spanning subgraph. A $k$-factor is a spanning $k$-regular subgraph.

Every regular bipartite graph has a $1$-factor.

*Not* every regular graph has a $1$-factor.

But...

**Theorem.** (Petersen, 1891) Every $2k$-regular graph has a $2$-factor.

*Proof.* Use Eulerian cycle of $G$ to create an auxiliary $k$-regular bipartite graph $H$, such that a perfect matching in $H$ corresponds to a $2$-factor in $G$.