How to find a maximum weight matching in a bipartite graph?

In the maximum weighted matching problem a non-negative weight $w_{i,j}$ is assigned to each edge x_iy_j of $K_{n,n}$ and we seek a perfect matching M to maximize the total weight $w(M) = \sum_{e \in M} w(e)$.

With these weights, a (weighted) cover is a choice of labels $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, such that $u_i + v_j \geq w_{i,j}$ for all i, j. The cost $c(u, v)$ of a cover (u, v) is $\sum u_i + \sum v_j$. The minimum weighted cover problem is that of finding a cover of minimum cost.

Duality Lemma For a perfect matching M and a weighted cover (u, v) in a bipartite graph G, $c(u, v) \geq w(M)$. Also, $c(u, v) = w(M)$ iff M consists of edges x_iy_j such that $u_i + v_j = w_{i,j}$. In this case, M and (u, v) are both optimal.

The algorithm

The equality subgraph $G_{u,v}$ for a weighted cover (u, v) is the spanning subgraph of $K_{n,n}$ whose edges are the pairs x_iy_j such that $u_i + v_j = w_{i,j}$. In the cover, the excess for i, j is $u_i + v_j - w_{i,j}$.

Hungarian Algorithm

Input. A matrix $(w_{i,j})$ of weights on the edges of $K_{n.n}$ with partite sets X and Y.

Idea. Iteratively adjusting a cover (u, v) until the equality subgraph $G_{u,v}$ has a perfect matching.

Initialization. Let $u_i = \max\{w_{i,j} : j = 1, \ldots, n\}$ and $v_j = 0$.

Iteration.

Form $G_{u,v}$ and find a maximum matching M in it.
IF M is a perfect matching, THEN
  stop and **report** M as a maximum weight matching and (u, v) as a minimum cost cover
ELSE
  let Q be a vertex cover of size $|M|$ in $G_{u,v}$.
  $R := X \cap Q$
  $T := Y \cap Q$
  $\epsilon := \min\{u_i + v_j - w_{i,j} : x_i \in X \setminus R, y_j \in Y \setminus T\}$
  Update u and v:
    $u_i := u_i - \epsilon$ if $x_i \in X \setminus R$
    $v_j := v_j + \epsilon$ if $y_j \in T$
Iterate

Theorem The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover.

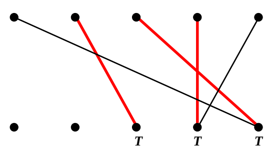
The Assignment Problem — An example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 7 & 2 \\ 1 & 3 & 4 & 4 & 5 \\ 3 & 6 & 2 & 8 & 7 \\ 4 & 1 & 3 & 5 & 4 \end{pmatrix}$$

Excess Matrix

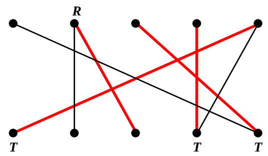
$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \\ \begin{array}{c} 5 \\ 8 \\ 5 \\ 8 \\ 5 \end{array} & \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 & 6 \\ 4 & 2 & 1 & 1 & 0 \\ 5 & 2 & 6 & 0 & 1 \\ 1 & 4 & 2 & 0 & 1 \end{pmatrix} \\ & \begin{array}{ccccc} & & T & T & T \end{array} \end{array}$$

Equality Subgraph



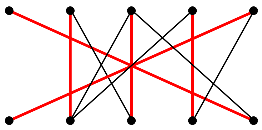
$\epsilon = 1$

$$
\begin{array}{c|ccccc|c}
 & 0 & 0 & 1 & 1 & 1 & \\
\hline
4 & 3 & 2 & 2 & 1 & 0 & \\
7 & 1 & 0 & 0 & 1 & 6 & R \\
4 & 3 & 1 & 1 & 1 & 0 & \\
7 & 4 & 1 & 6 & 0 & 1 & \\
4 & 0 & 3 & 2 & 0 & 1 & \\
\hline
 & T & & & T & T &
\end{array}
$$



$\epsilon = 1$

$$
\begin{array}{c|ccccc|}
 & 1 & 0 & 1 & 2 & 2 \\
\hline
3 & 3 & 1 & 1 & 1 & 0 \\
7 & 2 & 0 & 0 & 2 & 7 \\
3 & 3 & 0 & 0 & 1 & 0 \\
6 & 4 & 0 & 5 & 0 & 1 \\
3 & 0 & 2 & 1 & 0 & 1 \\
\hline
\end{array}
$$



DONE!!

The Duality Lemma states that if $w(M) = c(u, v)$ for some cover (u, v), then M is maximum weight.

We found a maximum weight matching (transversal). The fact that it is maximum is certified by the indicated cover, which has the same cost:

$$
\begin{array}{c|ccccc|}
 & 1 & 0 & 1 & 2 & 2 \\
\hline
3 & 1 & 2 & 3 & 4 & 5 \\
7 & 6 & 7 & 8 & 7 & 2 \\
3 & 1 & 3 & 4 & 4 & 5 \\
6 & 3 & 6 & 2 & 8 & 7 \\
3 & 4 & 1 & 3 & 5 & 4 \\
\hline
\end{array}
$$

$$
\begin{aligned}
w(M) &= 5+7+4+8+4 = 28 = \\
&= 1+0+1+2+2+ \\
&\quad 3+7+3+6+3 = c(u,v)
\end{aligned}
$$

Hungarian Algorithm — Proof of correctness

Proof. If the algorithm ever terminates and $G_{u,v}$ is the equality subgraph of a (u, v), which is indeed a cover, then M is a m.w.m. and (u, v) is a m.c.c. by Duality Lemma.

Why is (u, v), created by the iteration, a cover?

Let $x_i y_j \in E(K_{n,n})$. Check the four cases.

$x_i \in R$,	$y_j \in Y \setminus T$	$\Rightarrow$	u_i and v_j do not change.
$x_i \in R$,	$y_j \in T$	$\Rightarrow$	u_i does not change v_j increases.
$x_i \in X \setminus R$,	$y_j \in T$	$\Rightarrow$	u_i decreases by ϵ, v_j increases by ϵ.
$x_i \in X \setminus R$,	$y_j \in Y \setminus T$	$\Rightarrow$	$u_i + v_j \geq w_{i,j}$ by definition of ϵ.

Why does the algorithm terminate?

M is a matching in the new $G_{u,v}$ as well. So either
(i) max matching gets larger or
(ii) $\#$ of vertices reached from U by M-alternating paths grows. (U is the set of unsaturated vertices of M in X.)

Matchings in general graphs

An odd component is a connected component with an odd number of vertices. Denote by $o(G)$ the number of odd components of a graph G.

Theorem. (Tutte, 1947) A graph G has a perfect matching iff $o(G - S) \leq |S|$ for every subset $S \subseteq V(G)$.

Proof.

$\Rightarrow$ Easy.

$\Leftarrow$ (Lovász, 1975) Consider a counterexample G with the maximum number of edges.

Claim. $G + xy$ has a perfect matching for any $xy \notin E(G)$.

Proof of Tutte's Theorem — Continued

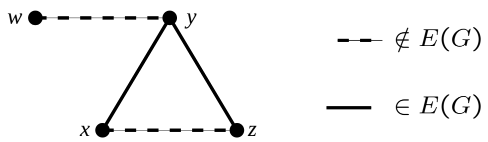
Define $U := \{v \in V(G) : d_G(v) = n(G) - 1\}$

Case 1. $G - U$ consists of disjoint cliques.

Proof: Straightforward to construct a perfect matching of G.

Case 2. $G - U$ is not the disjoint union of cliques.

Proof: Derive the existence of the following subgraph.



Obtain contradiction by constructing a perfect matching M of G using perfect matchings M_1 and M_2 of $G+xz$ and $G + yw$, respectively.

Corollaries

Corollary. (Berge,1958) For a subset $S \subseteq V(G)$ let $d(S) = o(G - S) - |S|$. Then

$$2\alpha'(G) = \min\{n - d(S) : S \subseteq V(G)\}.$$

Proof. $(\leq)$ Easy.

$(\geq)$ Apply Tutte's Theorem to $G \vee K_d$.

Corollary. (Petersen, 1891) Every 3-regular graph with no cut-edge has a perfect matching.

Proof. Check Tutte's condition. Let $S \subseteq V(G)$.
Double-count the number of edges between an S and the odd components of $G - S$.
Observe that between any odd component and S there are at least three edges.

Factors

A factor of a graph is a spanning subgraph. A k-factor is a spanning k-regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

Theorem. (Petersen, 1891) Every $2k$-regular graph has a 2-factor.

Proof. Use Eulerian cycle of G to create an auxiliary k-regular bipartite graph H, such that a perfect matching in H corresponds to a 2-factor in G.