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Graph Theory — Problem Set 5

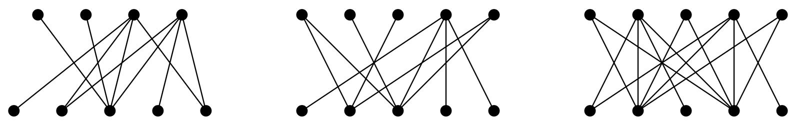
Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT03/`

Due Date: November 26, 2003 at the lecture

Exercise 5.1

(Exercise 3.1.1 in the Textbook)

(−) Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.



Exercise 5.2

(Exercise 3.1.7 in the Textbook)

(−) Prove that a graph G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G with no isolated vertices.

Exercise 5.3

(Exercise 3.1.8 in the Textbook)

(!) Prove or disprove: Every tree has at most one perfect matching.

Exercise 5.4

(Exercise 3.1.9 in the Textbook)

(!) Prove that every **maximal** matching in a graph G has at least $\alpha'(G)/2$ edges.

Exercise 5.5

(Exercise 3.1.19 in the Textbook)

(!) Let $\mathcal{A} = (A_1, \ldots, A_m)$ be a collection of subsets of a set Y. A **system of distinct representatives** (SDR) for $\mathcal{A}$ is a set of distinct elements $a_1, \ldots, a_m$ in Y such that $a_i \in A_i$ for every $i = 1, \ldots, m$. Prove that $\mathcal{A}$ has an SDR if and only if $|\bigcup_{i \in S} A_i| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$. (Hint: Transform this to a graph problem.)

Exercise 5.6

(Exercise 3.1.24 in the Textbook)

(!) A **permutation matrix** P is a $0,1$-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all row sums and column sums equal k.

Exercise 5.7

(Exercise 3.1.32 in the Textbook)

(!) In an X,Y-bigraph G (namely, a bipartite graph with X and Y as its partite sets), the **deficiency** of a set S is $\operatorname{def}(S) = |S| - |N(S)|$; note that $\operatorname{def}(\emptyset) = 0$. Prove that

$$\alpha'(G) = |X| - \max_{S \subseteq X} \operatorname{def}(S).$$

(Hint: Form a bipartite graph G' such that G' has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.) (Ore [1955])