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Graph Theory — Problem Set 6

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT03/`

Due Date: December 3, 2003 at the lecture

Exercise 6.1

(Exercise 3.2.5–6 in the Textbook)

a) Find a transversal of maximum total sum (weight) in the matrix (a) below. Prove that there is no larger weight transversal by exhibiting a solution to the dual problem. Explain why this proves that there is no larger transversal.

b) Find a *minimum-weight* transversal in the matrix (b) below, and use duality to prove that the solution is optimal. (Hint: Use a transformation of the problem.)

(a)

4	4	4	3	6
1	1	4	3	4
1	4	5	3	5
5	6	4	7	9
5	3	6	8	3

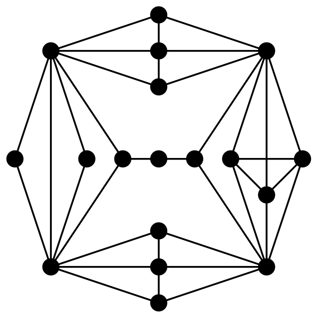
(b)

4	5	8	10	11
7	6	5	7	4
8	5	12	9	6
6	6	13	10	7
4	5	7	9	8

Exercise 6.2

(Exercise 3.3.2 in the Textbook)

Exhibit a maximum matching in the graph below, and give a short proof that it has no larger matching.



Exercise 6.3

(Exercise 3.1.29 in the Textbook)

(!) Use the König–Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Exercise 6.4

(Exercise 3.1.42 in the Textbook)

(!) An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G. (Caro [1979], Wei [1981])

To be continued on the back side...

Exercise 6.5

(Exercise 3.1.25 in the Textbook)

(!) A **doubly stochastic matrix** Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed $Q = c_1P_1 + \cdots c_mP_m$, where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. (See Exercise 5.6 for the definition of a permutation matrix.) For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(Hint: Use induction on the number of nonzero entries in Q.) (Birkhoff [1946], von Neumann [1953])

Exercise 6.6

(Exercise not in the Textbook)

(+) Here is a story to shed some light on how hard life was in the early 70s, in a kindergarten, in communist Hungary. There were 30 kids in my class. The inconsiderate regime of communist oppressors provided only 60 toys for us to play with. Every morning everybody could name a few of those toys which (s)he wanted to play with. It was always a big problem to distribute the toys, in such a way that everyone got two favorable ones.

One day my favorite kindergarten teacher, *Piri néni*, observed that for any group S of kids the number of toys which were chosen by at least one kid from S was at least $2|S|$. Then, she translated the problem into the language of graph theory, and finally she could assign the toys to the kids, such that every kid would get two toys (s)he favored.

What was her argument?