Eidgenössische Technische Hochschule Zürich

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Graph Theory — *Problem Set 8*

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT03/`

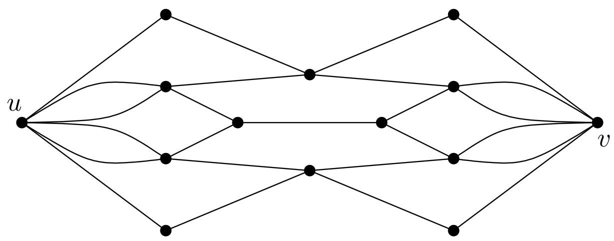
Due Date: December 17, 2003 at the lecture

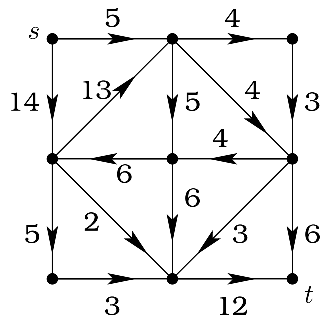
Exercise 8.1

(Exercise 4.2.1, 4.3.2 in the Textbook)

a) (–) Determine $\kappa(u, v)$ and $\kappa'(u, v)$ in the graph drawn left below. (Hint: Use the dual problems to give short proofs of optimality.)

b) (–) In the network right below, find a maximum flow from s to t. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.





Exercise 8.2

(Exercise 4.2.8 in the Textbook)

Prove that a simple graph G is 2-connected if and only if for every ordered triple (x, y, z) of distinct vertices, G has an x, z-path through y. (Chien [1968])

Exercise 8.3

(Exercise 4.2.12 in the Textbook)

(!) Use Menger's Theorem to prove that $\kappa(G) = \kappa'(G)$ when G is 3-regular.

Exercise 8.4

(Exercise 4.2.22 in the Textbook)

(!) Suppose that $\kappa(G) = k$ and $\operatorname{diam} G = d$. Prove that $n(G) \geq k(d-1)+2$ and $\alpha(G) \geq \lceil (1+d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph for which equality holds in both bounds. (The use of Menger's theorem is permitted.)

Exercise 8.5

(Exercise 4.2.23 in the Textbook)

(!) Use Menger's Theorem to prove the König–Egerváry Theorem ($\alpha'(G) = \beta(G)$ when G is bipartite).

To be continued on the back side...

Exercise 8.6

(Exercise scattered in the Textbook)

An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$. A **tournament** is an orientation of a complete graph. A **king** in a digraph is a vertex from which every vertex is reachable by a path of length at most 2.

a) (!) Prove that in a tournament every vertex of maximum out-degree is a king.

b) Let D be a tournament having no vertex with in-degree 0. By Part (a), we know that there is a king in a tournament. Prove that if x is a king in D, then D has another king in $N^{-}(x)$.

c) (+) Let x be a vertex of maximum out-degree in a tournament D. Prove that D has a spanning directed tree rooted at x (i.e., an orientation of a spanning tree where x has in-degree 0) such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity.) (Lu [1996])