Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Dr. Tibor Szabó and Yoshio Okamoto

December 17, 2003

Graph Theory — Problem Set 9

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT03/`

Due Date: January 7, 2004 at the lecture

Exercise 9.1

(Exercise 4.2.21 in the Textbook)

(!) Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Nash-Williams [1960])

Exercise 9.2

(Exercise 4.2.29 in the Textbook)

Given a graph G, let D be the digraph obtained by replacing each edge with two oppositely-directed edges having the same endpoints (thus D is the symmetric digraph with underlying graph G). Assume that, for all $x, y \in V(D)$, $\kappa_D(x, y) = \lambda_D(x, y)$ holds when $(x, y) \notin E(D)$. Use this hypothesis to prove that also $\kappa_G(x, y) = \lambda_G(x, y)$ for $\{x, y\} \notin E(G)$.

Exercise 9.3

(Exercise not in the Textbook)

In the eighth grade, playing in the little soccer league, our school (*Áldás*) was in a fierce competition with four others: the *Ady*, the *Fillér*, the *Medve*, and the *Törökvészi* elementary schools. During the year every team played every other team 6 times. Despite my brilliant effort as right mid-fielder, in the middle of the season we were standing without a single win, with twelve losses. Our coach was still very enthusiastic and in order to motivate us, got into some complicated argument about how we can still win at least a share of the championship.

The situation was the following. Ady had 5 more games with Fillér, 2 more games with Medve, and 5 with Törökvészi. Fillér had 3 more games with Medve, and 6 with Törökvészi. Medve had to play 3 more times with Törökvészi. Ady was at first place with 8 wins, Törökvészi was second with 7, Fillér had 6, and even the much despised Medve was ahead of us with 3 wins.

Was our coach right about his calculations or did he just want to fire us up before our usual showdown with Medve? (Hint: you can try to model the problem as a network flow.)

Exercise 9.4

(Exercise not in the Textbook)

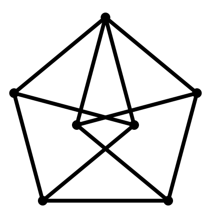
(+) Let $\mathcal{A} = \{A_1, \dots, A_m\}$ and $\mathcal{B} = \{B_1, \dots, B_m\}$ be collections of subsets of a set Y. A **common system of distinct representatives** (CSDR) for $\mathcal{A}$ and $\mathcal{B}$ is a set of distinct elements $x_1, \dots, x_m$ in Y which is both an SDR for $\mathcal{A}$ and an SDR for $\mathcal{B}$. (For the definition of an SDR, see Exercise 5.5.) Prove that there exists a CSDR for $\mathcal{A}$ and $\mathcal{B}$ if and only if

$$\left| \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) \right| \geq |I| + |J| - m \quad \text{for each pair } I, J \subseteq [m].$$

Exercise 9.5

(Exercise 5.1.1 in the Textbook)

(−) Compute the clique number, the independence number and the chromatic number of the graph below. Is the graph color-critical?



Exercise 9.6

(Exercise 5.1.20 in the Textbook)

(!) Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

Exercise 9.7

(Exercise 5.1.41 in the Textbook)

(!) Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Nordhaus–Gaddum [1956])

Have a nice Christmas and a happy new year!