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Graph Theory — Problem Set 11

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT03/`

Due Date: January 21, 2004 at the lecture

Exercise 11.1

(Exercise 5.2.2 in the Textbook)

(−) Prove that a simple graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge (i.e., $K_1 + K_2$).

Exercise 11.2

(Exercise 5.2.15 in the Textbook)

(!) Prove that every triangle-free n-vertex graph has chromatic number at most $2\sqrt{n}$. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)

Exercise 11.3

(Exercise not in the Textbook)

a) Prove that the Turán graph $T_{n,r-1}$ is a unique graph which maximizes the sum of the squared degrees (i.e., $\sum_{v\in V(G)} d(v)^2$) among all K_r-free n-vertex graphs G. (Hint: Mimic the proof of Turán's theorem.)

b) Prove that the statement of part (a) is no longer generally true if we consider maximizing $\sum_{v\in V(G)} d(v)^4$ instead.

Exercise 11.4

(Exercise not in the Textbook)

Let s, t, n be natural numbers such that $0 < s \le t \le n$.

a) Let G be a graph with n vertices which does not contain any $K_{t,s}$ as a subgraph. Prove that $\sum_{v\in V(G)} \binom{d(v)}{s} \le (t-1)\binom{n}{s}$. (Hint: Count the number of copies of $K_{1,s}$ in two ways.)

b) Use part (a) to prove that $ex(n, K_{t,s}) \le Cn^{2-1/s}$ for some constant C depending only on s, t. (Hint: Use the estimates $(\frac{a}{b})^b \le \binom{a}{b} \le a^b$ and Jensen's inequality.)

Exercise 11.5

(Exercise not in the Textbook)

a) Given n distinct points in the plane, prove that the distance is exactly 1 for at most $O(n^{3/2})$ pairs. (Hint: First prove that the unit-distance graph contains no $K_{3,2}$, and apply the result in Exercise 11.4.)

b) Given n distinct points in the 3-dimensional space, prove that the distance is exactly 1 for at most $O(n^{5/3})$ pairs.

Exercise 11.6

(Exercise not in the Textbook)

Let p be a prime number, and $\mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$. Consider the following graph G_p. The vertex set of G_p is $\mathbb{F}_p^2 \setminus \{(0,0)\}$, and an edge is drawn between distinct $(a,b), (a',b') \in \mathbb{F}_p^2 \setminus \{(0,0)\}$ if and only if $aa' + bb' \equiv 1 \mod p$.

a) Prove that G_p does not contain $K_{2,2}$. (Hint: You can utilize the fact that $\mathbb{F}_p$ constitutes a field under the addition and the multiplication modulo p.)

b) Show that $e(G_p) \ge (p-1)(p^2-1)/2$.

c) From parts (a) and (b), conclude that $ex(n, K_{2,2}) = \Omega(n^{3/2})$.

(Comment: Together with Exercise 11.4, we can see that $ex(n, K_{2,2}) = \Theta(n^{3/2})$.)

Please look at the back side for the supplement.

Supplement

1. Convex functions and Jensen's inequality

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if for any $x, y \in \mathbb{R}^n$ and for any $\lambda \in [0, 1]$

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

A convex function f satisfies the following inequality: For any integer $k \geq 1$, any $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, and any $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ with $\sum_{i=1}^{k} \lambda_i = 1$, it holds that

$$\sum_{i=1}^{k} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{k} \lambda_i x_i\right).$$

This inequality is called **Jensen's inequality**.

2. Big-O notation

Let $f, g : \mathbb{N} \to \mathbb{R}$ be two functions. We write $f(n) = O(g(n))$ if f is bounded by g from above in the order of magnitude. Formally speaking, we say $f(n) = O(g(n))$ if there exist k and M (which depend on f, g) such that for all $n > k$ it holds that $|f(n)/g(n)| \leq M$. If $g(n) = O(f(n))$, we write $f(n) = \Omega(g(n))$. If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then we write $f(n) = \Theta(g(n))$.

Another frequently encountered notation is the little-o. We write $f(n) = o(g(n))$ if f is negligible compared to g in the order of magnitude (or f is less than g in the order of magnitude). Formally speaking, we say $f(n) = o(g(n))$ if $\lim_{n\to\infty} |f(n)/g(n)| = 0$.

Here is the summary.

Notation	Definition	Interpretation
$f(n) = O(g(n))$	$\exists k, M\ \forall n > k$: $\lvert f(n)/g(n)\rvert \leq M$	f is at most g in the order of magnitude.
$f(n) = \Omega(g(n))$	$\exists k, M\ \forall n > k$: $\lvert g(n)/f(n)\rvert \leq M$	f is at least g in the order of magnitude.
$f(n) = \Theta(g(n))$	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$	f has the same order of magnitude as g.
$f(n) = o(g(n))$	$\lim_{n\to\infty} \lvert f(n)/g(n)\rvert = 0$	f is less than g in the order of magnitude.