Chapter 10

Crossings

So far within this course we have mostly tried to avoid edge crossings and studied classes of planar graphs that allow us to avoid crossings altogether. However, without doubt there are many interesting graphs that are not planar, and still we would like to draw them in a reasonable fashion. An obvious quantitative approach is to still avoid crossings as much as possible, even if they cannot be avoided completely.

For a graph $G = (V, E)$, the *crossing number* $\mathrm{cr}(G)$ is defined as the minimum number of edge crossings over all possible drawings of G. In an analogous fashion, the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ is defined as the minimum number of edge crossings over all possible straight-line drawings of G.

In order to see that thes notions are well-defined, let us first argue that the number of crossings in a minimum-crossing drawing is finite and, in fact, upper bounded by $\binom{|E|}{2}$.

Lemma 10.1. *In a drawing of a graph G with $\mathrm{cr}(G)$ crossings, every two distinct edges share at most one point.*

Proof. By a rerouting argument... □

In particular, by Lemma 10.1 no two adjacent edges (that is, edges that have a common endpoint) cross. A drawing that satisfies the statement of Lemma 10.1 is called a *simple topological drawing*. So, using this term, Lemma 10.1 could also be stated as "Every minimum-crossing drawing is a simple topological drawing."

It is quite easy to give an upper bound on the crossing number of a particular graph, simply by describing a drawing and counting the number of crossings in that drawing. Conversely, it is much harder to give a lower bound on the crossing number of a graph because such a bound corresponds to a statement about *all* possible drawings of that graph. But the following simple lower bound can be obtained by counting edges.

Lemma 10.2. *For a graph G with $n \geqslant 3$ vertices and e edges, we have $\mathrm{cr}(G) \geqslant e - (3n - 6)$.*

Proof. Consider a drawing of $G = (V, E)$ with $\mathrm{cr}(G)$ crossings. For each crossing, pick one of the two involved edges arbitrarily. Obtain a new graph $G' = (V, E')$ from G by removing all picked edges. By construction G' is plane and, therefore, $|E'| \leqslant 3n-6$ by Corollary 2.5. As at most $\mathrm{cr}(G)$ edges were picked (some edge could be picked for several crossings), we have $|E'| \geqslant |E| - \mathrm{cr}(G)$. Combining both bounds completes the proof. □

The bound in Lemma 10.2 is quite good if the number of edges is close to $3n$ but not so good for dense graphs. For instance, for the complete graph K_n the lemma guarantees a quadratic number of crossings, whereas according to the Harary-Hill Conjecture [?]

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \in \Theta(n^4).$$

So for a dense graph G we should try a different approach. Given that the bound in Lemma 10.2 is not so bad for sparse graphs, why not apply it to some sparse subgraph of G? This astonishingly simple idea turns out to work really well, as the following theorem demonstrates.

Theorem 10.3 (Crossing Lemma [?]). *For a graph G with n vertices and $e \geqslant 4n$ edges, we have $\mathrm{cr}(G) \geqslant e^3/(64n^2)$.*

Proof. Consider a drawing of G with $\mathrm{cr}(G)$ crossings. Take a random induced subgraph of G by selecting each vertex independently with probability p (a suitable value for p will be determined later). By this process we obtain a random subset $U \subseteq V$ and the corresponding induced subgraph $G[U]$, along with an induced drawing for $G[U]$. Consider the following three random variables:

- N, the number of vertices selected, with $E[N] = pn$;
- M, the number of edges induced by the selected vertices, with $E[M] = p^2e$; and
- C, the number of crossings induced by the selected vertices and edges, with $E[C] = p^4\mathrm{cr}(G)$. (Here we use that adjacent edges do not cross in a minimum-crossing drawing by Lemma 10.1.)

According to Lemma 10.2, these quantities satisfy $C \geqslant \mathrm{cr}(G[U]) \geqslant M - 3N$. Taking expectations on both sides and using linearity of expectation yields $E[C] \geqslant E[M] - 3E[N]$ and so $p^4\mathrm{cr}(G) \geqslant p^2e - 3pn$. Setting $p = 4n/e$ (which is $\leqslant 1$ due to the assumption $e \geqslant 4n$) gives

$$\mathrm{cr}(G) \geqslant \frac{e}{p^2} - 3\frac{n}{p^3} = \frac{e^3}{16n^2} - 3\frac{e^3}{64n^2} = \frac{e^3}{64n^2}$$

□

The constant $1/64$ in the statement of Theorem 10.3 is not the best possible. On one hand, Ackerman [?] showed that $1/64$ can be replaced by $1/29$, at the cost of requiring $e \geqslant 7n$. On the other hand, Pach and Tóth [?] describe graphs with $n \ll e \ll n^2$ that have crossing number at most

$$\frac{16}{27\pi^2}\frac{e^3}{n^2} < \frac{1}{16.65}\frac{e^3}{n^2}.$$

Hence it is not possible to replace $1/64$ by $1/16.65$ in the statement of Theorem 10.3.

In the remainder of this chapter, we will discuss several nontrivial bounds on the size of combinatorial structures that can be obtained by a more-or-less straightforward application of the Crossing Lemma. These beautiful connections were observed by Székely [?], the original proofs were different and more involved.

Theorem 10.4 (Szemerédi-Trotter [?]). *The maximum number of incidences between n points and m lines in $\mathbb{R}^2$ is at most $\sqrt[3]{32} \cdot n^{2/3}m^{2/3} + 4n + m$.*

Proof. Let P denote the given set of n points, and let L denote the given set of m lines. We may suppose that every line from L contains at least one point from P. (Discard all lines that do not; they do not contribute any incidence.) Denote by I the number of incidences between P and L. Consider the graph $G = (P, E)$ whose vertex set is P, and where a pair p, q of points is connected by an edge if p and q appear consecutively along some line $\ell \in L$ (that is, both p and q are incident to ℓ and no other point from P lies on the line segment $\overline{pq}$). Using the straight-line drawing induced by the arrangement of L we may regard G as a geometric graph with at most $\binom{m}{2}$ crossings.

Every line from L contains $k \geqslant 1$ point(s) from P and contributes $k-1$ edges to G. Hence $|E| = I - m$. If $|E| \leqslant 4n$, then $I \leqslant 4n + m$ and the theorem holds. Otherwise, we can apply the Crossing Lemma to obtain

$$\binom{m}{2} \geqslant \mathrm{cr}(G) \geqslant \frac{|E|^3}{64n^2}$$

and so $I \leqslant \sqrt[3]{32}\, n^{2/3}m^{2/3} + m$. □

Theorem 10.5. *The maximum number of unit distances determined by n points in $\mathbb{R}^2$ is at most $5n^{4/3}$.*

Proof. Let P denote the given set of n points, and consider the set C of n unit circles centered at the points in P. Then the number I of incidences between P and C is exactly twice the number of unit distances between points from P.

Define a graph $G = (P, E)$ on P where two vertices p and q are connected by an edge if they appear consecutively along some circle $c \in C$ (that is, $p, q \in c$ and at least one of the circular arcs of c between p and q does not contain any other point from P). Clearly $|E| = I$.

Obtain a new graph $G' = (P, E')$ from G by removing all edges along circles from C that contain at most two points from P. Note that $|C| = n$ and that every circle whose edges are removed contributes at most two edges to G. Therefore $|E'| \geqslant |E| - 2n$. In G' there are no loops and no two vertices are connected by two edges along the same circle. Therefore, any two vertices are connected by at most two edges because there are exactly two distinct unit circles passing through any two distinct points in $\mathbb{R}^2$.

Obtain a new graph $G'' = (P, E'')$ from G' by removing one copy of every double edge. Clearly G'' is a simple graph with $|E''| \geqslant |E'|/2 \geqslant (|E|/2) - n$. As every pair of circles intersects in at most two points, we have $\mathrm{cr}(G'') \leqslant 2\binom{n}{2} \leqslant n^2$.

If $|E''| \leqslant 4n$, then $(|E|/2) - n \leqslant 4n$ and so $I = |E| \leqslant 10n < 10n^{4/3}$ and the theorem holds. Otherwise, by the Crossing Lemma we have

$$n^2 \geqslant \operatorname{cr}(G'') \geqslant \frac{|E''|^3}{64n^2}$$

and so $|E''| \leqslant 4n^{4/3}$. It follows that $I = |E| \leqslant 8n^{4/3} + 2n \leqslant 10n^{4/3}$. □

Theorem 10.6. *For $A \subset \mathbb{R}$ with $|A| = n \geqslant 3$ we have $\max\{|A+A|, |A \cdot A|\} \geqslant \frac{1}{4}n^{5/4}$.*

Proof. Let $A = \{a_1, \ldots, a_n\}$. Set $X = A + A$ and $Y = A \cdot A$. We will show that $|X||Y| \geqslant \frac{1}{16}n^{5/2}$, which proves the theorem. Let $P = X \times Y \subset \mathbb{R}^2$ be the set of points whose x-coordinate is in X and whose y-coordinate is in Y. Clearly $|P| = |X||Y|$. Next define a set L of lines by $\ell_{ij} = \{(x, y) \in \mathbb{R}^2 : y = a_i(x - a_j)\}$, for $i, j \in \{1, \ldots, n\}$. Clearly $|L| = n^2$.

On the one hand, every line ℓ_{ij} contains at least n points from P because for $x_k = a_j + a_k \in X$ and $y_k = a_i(x_k - a_j) = a_i a_k \in Y$ we have $(x_k, y_k) \in P \cap \ell_{ij}$, for $k \in \{1, \ldots, n\}$. Therefore the number I of incidences betwen P and L is at least n^3.

On the other hand, by the Szemerédi-Trotter Theorem we have $I \leqslant \sqrt[3]{32}|P|^{2/3}n^{4/3} + 4|P| + n^2$. Combining both bounds we obtain

$$n^3 \leqslant \sqrt[3]{32}|P|^{2/3}n^{4/3} + 4|P| + n^2.$$

Hence either $4|P| + n^2 \geqslant \frac{n^3}{2}$, which implies $|P| \geqslant \frac{1}{16}n^{5/2}$, for $n \geqslant 3$; or $\sqrt[3]{32}\,|P|^{2/3}n^{4/3} \geqslant \frac{n^3}{2}$ and thus

$$|P|^{2/3} \geqslant \frac{n^3}{2\sqrt[3]{32}\,n^{4/3}} = \left(\frac{n^5}{256}\right)^{1/3} \implies |P| \geqslant \frac{n^{5/2}}{16}.$$

□

Exercise 10.7. *Consider two edges e and f in a topological plane drawing so that e and f cross at least twice. Prove or disprove: There exist always two distinct crossings x and y of e and f so that the portion of e between x and y is not crossed by f and the portion of f between x and y is not crossed by e.*

Exercise 10.8. *Let G be a graph with $n \geqslant 3$ vertices, e edges, and $\operatorname{cr}(G) = e - (3n - 6)$. Show that in every drawing of G with $\operatorname{cr}(G)$ crossings, every edge is crossed at most once.*

Exercise 10.9. *Consider the abstract graph G that is obtained as follows: Start from a plane embedding of the 3-dimensional (hyper-)cube, and add in every face a pair of (crossing) diagonals. Show that $\operatorname{cr}(G) = 6 < \overline{\operatorname{cr}}(G)$.*

Exercise 10.10. *A graph is* 1-planar *if it can be drawn in the plane so that every edge is crossed at most once. Show that a 1-planar graph on $n \geqslant 3$ vertices has at most $4n - 8$ edges.*

Exercise 10.11. *Show that the bound from the Crossing Lemma is asymptotically tight: There exists a constant* c *so that for every* $n, e \in \mathbb{N}$ *with* $e \leqslant \binom{n}{2}$ *there is a graph with* n *vertices and* e *edges that admits a plane drawing with at most* ce^3/n^2 *crossings.*

Exercise 10.12. *Show that the maximum number of unit distances determined by* n *points in* $\mathbb{R}^2$ *is* $\Omega(n \log n)$*. Hint: Consider the hypercube.*

Questions

52. *What is the crossing number of a graph? What is the rectilinear crossing number?* Give the definitions and examples. Explain the difference.
53. *For a nonplanar graph, the more edges it has, the more crossings we would expect. Can you quantify such a correspondence more precisely?* State and prove Lemma 10.2 and Theorem 10.3 (The Crossing Lemma).
54. *Why is it called "Crossing Lemma" rather than "Crossing Theorem"?* Explain at least two applications of the Crossing Lemma, for instance, your pick out of the theorems 10.4, 10.5, and 10.6.