Chapter 11

Counting

We consider the problems of (i) counting the number of simplices spanned by $d+1$ points in a finite set of n points in d-space that contain a given point (*simplicial depth*) and (ii) counting the *number of facets of the convex hull* of n points in d-space. These problems are closely related by a duality (different from what has been dealt with before) called *Gale Duality*.

Counting refers to *algorithmic counting* (given an input, determine the number of certain objects) and to *extremal counting* (given all configurations with certain numbers, what are the extremal number of objects that can occur). Sometimes, one considers also *enumeration* (given an input, produce all objects under consideration).

Notation. Here are a few notational conventions: $0 := (0, 0, \ldots, 0)$ is the origin in the considered ambient space $\mathbb{R}^d$; $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\text{conv}(S)$ denotes the convex hull of a given set S in $\mathbb{R}^d$; $\binom{S}{k}$ denotes the set of all k-element subsets of a given set S.

It will be useful to remember

$$\sum_{i=0}^{n-1} \binom{i}{k-1} = \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} .$$

It does not hurt to recapitulate the combinatorial interpretation of these identities: Recall that $\binom{n}{k}$ is the number of k-element subsets of an n-element set, i.e. $\binom{n}{k} = |\binom{[n]}{k}|$, for $[n] := \{1, 2, \ldots, n\}$.

Every set $A \in \binom{[n]}{k}$ has a maximum element. We charge[1] A to this maximum element. For $j \in [n]$ to be the maximum element in $A = \binom{[n]}{k}$, it has to be composed of j itself together with a $(k-1)$-element subset of $[j-1]$, of which there are exactly $\binom{j-1}{k-1}$ – this

[1]By the end of the day, this "charging" here is nothing else but a mapping, here $A \mapsto \max(A)$. In a similar situation, we will later actually say $\max(A)$ is a "witness" of A. This is established counting jargon, and often objects charge to (or witness) several elements.

is how often j is charged. Therefore,

$$\binom{n}{k} = \sum_{j=1}^{n} \binom{j-1}{k-1},$$

which proves the first equality. For the second equality, we discriminate the sets in $A \in \binom{[n]}{k}$ by whether the maximum element is n or not. There are $\binom{n-1}{k-1}$ sets where n is the maximum, and there are $\binom{n-1}{k}$ sets, where n is not the maximum. This shows

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Checkpoints are usually simple facts that are put there to check your understanding of definitions or notions, to be answered perhaps in a minute or two (assuming you have indeed absorbed the definition).

11.1 Introduction

Consider a set $P \subseteq \mathbb{R}^d$ and $q \in \mathbb{R}^d$. A set $A \in \binom{P}{d+1}$ is called q-*embracing simplex* if $q \in \text{conv}(A)$. The simplicial depth, $\text{sd}_q(P)$, of point q relative to P is the number of q-embracing simplices, i.e.

$$\text{sd}_q(P) := \left| \left\{ A \in \binom{P}{d+1} \;\middle|\; q \in \text{conv}(A) \right\} \right| .$$

This notion, among others, is a possible response to the search for a higher-dimensional counterpart of the notion of a median in $\mathbb{R}^1$. Note here that when specialized to $\mathbb{R}^1$, a median is a point of maximum simplicial depth. We will investigate here this notion, asking questions like:

> *What is the maximum possible simplicial depth a point can have in any set of* n *points in general position?*
>
> *How efficiently can we compute the simplicial depth of a point?*

A second question we want to address is that of the complexity of polytopes in general dimension d.

> *How many facets can a polytope obtained as the convex hull of* n *points have, how few?*
>
> *Given* n *points, how efficiently can we compute the number of facets of its convex hull? Can we do that asymptotically faster than enumerating these facets (which is a hard enough problem per se)?*

A small caveat, as has been mentioned in a previous chapter: We know that a 2-dimensional polytope with n vertices has n facets (here edges), and a 3-dimensional polytope with n vertices has at most $2n-4$ facets (if the vertices are in general position, it is exactly this number). So the number of facets is linear in n. This last fact is not true in higher dimension and we will see what the right bounds are. We will see that these two types of questions about simplicial depth and number of facets of a polytope are very closely related; in some sense, that we will make very explicit (via the so-called Gale Duality), it is the same question.

11.2 Embracing k-Sets in the Plane

In this section we investigate simplicial depth in the plane. First we generalize by considering arbitrarily large sets with a given point in their convex hull. Not only is this a natural step to take, but as we will see later, this extension to larger sets was unavoidable, even when we are interested in simplicial depth only.

Consider a set $P \subseteq \mathbb{R}^2$, with $0 \notin P$ and $P \,\dot\cup\, \{0\}$ in general position (no three on a line); $n := |P|$. This setting will be implicitly assumed throughout the section. For $k \in \mathbb{N}_0$, we define

$$e_k = e_k(P) := \left| \left\{ A \in \binom{P}{k} \;\middle|\; 0 \in \operatorname{conv}(A) \right\} \right| .$$

We call A with $0 \in \operatorname{conv}(A)$ an *embracing* k*-set*. For $|A| = 3$, we call A an *embracing triangle*.

Checkpoint 11.1. *e_3 is the simplicial depth of 0 in P, i.e. $e_3 = \mathrm{sd}_0(P)$. $e_0 = e_1 = e_2 = 0$, $e_n \in \{0, 1\}$ (where we use the general position assumption).*

We start a general investigation of the vector $\vec{e} = (e_0, e_1, \ldots, e_n) \in \mathbb{N}_0^{n+1}$. Bounds and algorithms will follow easily, but you need to be patient until it will be apparent, how everything fits nicely together – reward will come. For a preparatory step consider real vectors $\vec{x}_{0..n-3} = (x_0, x_1, \ldots, x_{n-3})$, $\vec{y}_{0..n-2}$ and $\vec{z}_{0..n-1}$ satisfying

$$e_k = \sum_{i=0}^{n-3} \binom{i}{k-3} x_i , \quad \text{for } k \geqslant 3, \tag{11.2}$$

$$e_k = \sum_{i=0}^{n-2} \binom{i}{k-2} y_i , \quad \text{for } k \geqslant 2, \text{ and} \tag{11.3}$$

$$e_k = \sum_{i=0}^{n-1} \binom{i}{k-1} z_i , \quad \text{for } k \geqslant 1. \tag{11.4}$$

Observe that $\vec{x}_{0..n-3}$ exists and is uniquely determined by $\vec{e}_{3..n}$, since

$$\begin{array}{rcll} e_n & = & \binom{n-3}{n-3} x_{n-3} & \Rightarrow x_{n-3} = e_n \\ e_{n-1} & = & \binom{n-4}{n-4} x_{n-4} + \binom{n-3}{n-4} x_{n-3} & \Rightarrow x_{n-4} = e_{n-1} - (n-3)\underbrace{x_{n-3}}_{e_n} \\ & \vdots & & \end{array}$$

Similarly, this works for $\vec{y}_{0..n-2}$ and $\vec{z}_{0..n-1}$. Thus we have

$$\vec{e}_{3..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} \vec{x}_{0..n-3}, \quad \vec{e}_{2..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} \vec{y}_{0..n-2}, \quad \vec{e}_{1..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} \vec{z}_{0..n-1} .$$

For now, it is by no means clear what that should help here. Note also that these facts are true for any vector e, we have not used any of the properties of the specific vector we are interested in. They simply describe one of many possible transformations of a given vector.

11.2.1 Adding a Dimension

Another step that comes across unmotivated: Lift the point set P vertically to a set P' in space, arbitrarily, with the only condition that P' is in general position (no four on a plane).[2] We denote the map by

$$P \ni \quad q = (x, y) \mapsto q' = (x, y, z(q)) \quad \in P' .$$

For an embracing triangle $\Delta = \{p, q, r\}$ in the plane, let β_Δ be the number of points in P' below the plane containing $\Delta' = \{p', q', r'\}$. (Just to avoid confusion: β_Δ clearly depends on the choice of the lifting P'.) Let

$$h_i = h_i(P') := \text{ the number of embracing triangles } \Delta \text{ with } \beta_\Delta = i.$$

Checkpoint 11.5. $\sum_{i=0}^{n-3} h_i = e_3$.

Let us recall here that we are assuming general position for $P \,\dot\cup\, \{0\}$.

Lemma 11.6. $0 \in \text{conv}(P) \iff h_0 = h_{n-3} = 1$.

Proof. ($\Leftarrow$) That's obvious, since $h_0 = 1$ means that there is some embracing triangle, and therefore 0 is in the convex hull of P.

($\Rightarrow$) Note that $0 \in \text{conv}(P)$ iff the z-axis (i.e. the vertical line through 0 in $\mathbb{R}^3$) intersects $\text{conv}(P')$. There are exactly two facets (triangles because of general position) intersected by the z-axis, the bottom one, let us call it Δ'_0, has no point in P' below its supporting plane, hence, $\beta_{\Delta_0} = 0$; the top one, Δ'_1, has no point in P' above and hence $n-3$ points in P' below (all but the three points defining the facet), hence, $\beta_{\Delta_1} = n-3$. Since any triple $\Delta' \subseteq P'$ with all points in P' on one side (above or below) must give rise to a facet, it cannot be hit by the z-axis, unless $\Delta' = \Delta'_0$ or $\Delta' = \Delta'_1$. □

[2]For example, choose the lifting map $(x, y) \mapsto (x, y, x^2 + y^2)$ ... but stay flexible!

Consider an embracing k-set A and its lifting A'. As observed before, in $\mathbb{R}^3$ the vertical line through 0 will intersect the boundary of $\text{conv}(A')$ in two facets. Consider the top facet – its vertices are liftings of some embracing triangle Δ in the plane. We call this Δ the *witness of* (the embracing property of) A. For how many embracing k-sets is Δ the witness?

For Δ to be witness of an embracing k-set B, we must have $\Delta \subseteq B$ and the remaining $k-3$ points in $B \setminus \Delta$ must be chosen so that $B' \setminus \Delta'$ lies below the plane spanned by Δ'. Hence Δ is witness for exactly $\binom{\beta_\Delta}{k-3}$ embracing k-sets. It follows that

$$e_k = \sum_{\Delta \text{ embracing}} \binom{\beta_\Delta}{k-3} = \sum_{i=0}^{n-3} \binom{i}{k-3} h_i \,. \tag{11.7}$$

That is, the h_i's are exactly the x_i's defined by equations (11.2). As observed before, we thus have

$$\vec{e}_{3..n} \overset{\text{determine}}{\underset{\text{each other}}{\longleftrightarrow}} \vec{h}_{0..n-3} := (h_0, h_1, \ldots, h_{n-3})$$

and therefore the vector $\vec{h}_{0..n-3}$ is independent of the lifting we chose, i.e. $h_i = h_i(P)$.

A few properties emerge. First note that $\vec{h}$ (consisting of nonnegative integers, each at most $\binom{n}{3} = O(n^3)$, i.e. $O(\log n)$ bits) is a compact way of representing $\vec{e}$ (with numbers, some may be exponential in n, with $\Omega(n)$ bits). Also, since it is easy to compute the vector $\vec{h}$ in $O(n^4)$ time[3], we can compute each entry of e_k in $O(n^4)$ time.

The independence of the vector $\vec{h}$ from the chosen lifting allows quite simple proofs of properties of $\vec{h}$: You can choose the lifting! If you can make a property of $\vec{h}$ hold for a chosen lifting, then it will be true for all liftings. Keep this in mind, when solving the following exercise.

Exercise 11.8. *Show*

$$h_0 = 1 \quad \Leftrightarrow \quad 0 \in \text{conv}(P) \quad \Leftrightarrow \quad h_i \geqslant 1 \textit{ for } 0 \leqslant i \leqslant n-3 \,.$$

Now, don't hesitate to use the assertion of this exercise and relation (11.7) for the following exercise.

Exercise 11.9. *Assume* $0 \in \text{conv}(P)$. *(i) What is the minimal possible value of* e_3 *in terms of* $n := |P|$*? (Note that this gives a quantified version of Carathéodory's Theorem.) (ii) What is the minimal possible value of* e_k*,* $3 \leqslant k \leqslant n$*?*

Exercise 11.10. *Let* P *be set of* n *points in general position (with the origin) in the plane. Consider the vector* $\vec{h}_{0..n-3}$ *as defined for a generic lifting in the lecture. What does* $\sum_{i=0}^{n-3} 2^i h_i$ *count?*

[3]With some basics from computational geometry, in $O(n^3)$ time.

Exercise 11.11. *Let* P *be set of* n *points in general position (with the origin) in the plane. Consider the numbers* e_k *as defined in the lecture for* P *(the number of embracing* k*-sets of* P*). Show* $\sum_{k=3}^{n}(-1)^k e_k = -1$ *provided* $0 \in \text{conv}(P)$.
(Hint: Plug in $\sum_{i=0}^{n-3}\binom{i}{k-3}h_i$ *for* e_k *in this sum and simplify, with the vector* $\vec{h}_{0..n-3}$ *as defined for a generic lifting in the lecture.)*

In a next step we show that the vector $\vec{h}$ is symmetric.

Lemma 11.12. $h_i = h_{n-3-i}$.

Proof. Define $\hat{h}_i$ in the same way as h_i, except that you count the points *above* (instead of below) the plane through the lifting of an embracing triangle. First, note that $\hat{h}_i = h_{n-3-i}$. And clearly, (with the same witness argument as before)

$$e_k = \sum_{i=0}^{n-3}\binom{i}{k-3}\hat{h}_i ,$$

and, therefore,

$$h_i = \hat{h}_i = h_{n-3-i} .$$

□

That is, vector $\vec{h}_{0..n-3}$ is determined by entries $h_0, h_1, \ldots, h_{\lfloor (n-3)/2 \rfloor}$.

Exercise 11.13. *Let* P *be set of* n *points in general position (with the origin) in the plane. Consider the numbers* e_k *as defined in the lecture for* P *(the number of embracing* k*-sets of* P*). Show* $(n-3)e_3 = 2e_4$.

Exercise 11.14. *Show that if* $|P| = 6$*, then* e_3 *determines* $e_{3..6}$*. How?*

Exercise 11.15. *Show that if* $|P|$ *is even then* e_3 *is even.*

11.2.2 The Upper Bound

We have seen in one of the exercises how the relation between $\vec{e}$ and $\vec{h}$ can be useful in proving lower bounds on the e_k's. We need two lemmas towards a proof of upper bounds.

The first lemma states that removing a point in P cannot increase h_j.

Lemma 11.16. *For all* $j \in \mathbb{N}_0$ *and all* $q \in P$, $h_j(P \setminus \{q\}) \leqslant h_j(P)$.

Proof. Which changes happen to h_j as we remove a point q in P?

- *We lose* embracing triangles Δ with j points below (in the lifting), where $q \in \Delta$ or q is one of the points below.
- *We keep* embracing triangles Δ with j points below, and q above.
- *We gain* embracing triangles Δ with $j+1$ points below, with q one of those below.

Now move q' (in the lifting) vertically above all planes defined by three points in $P' \setminus \{q'\}$. This does not change the values h_i (since, again, $\vec{h}$ is independent of the lifting), and the case of "We gain" cannot occur. This gives the lemma. □

Lemma 11.17. *For all* $j \in \mathbb{N}_0$, $\sum_{q \in P} h_j(P \setminus \{q\}) = (n-j-3)h_j(P) + (j+1)h_{j+1}(P)$.

Proof. A contribution to $\sum_{q \in P} h_j(P \setminus \{q\})$ can come only from triangles Δ with $\beta_\Delta = j$ or $\beta_\Delta = j+1$ (relative to the complete set P and a chosen lifting P').

- If $\beta_\Delta = j$, Δ' remains a triangle with j points below, if q is chosen as one of the $(n-3-j)$ points above.
- If $\beta_\Delta = j+1$, Δ' turns into a triangle with j points below, if q is chosen as one of the $(j+1)$ points below.

Hence the lemma. □

We recall the previous Lemma 11.16 to bound the sum in Lemma 11.17:

$$\sum_{q \in P} h_j(P \setminus \{q\}) \leqslant n \cdot h_j(P) \ ,$$

and with this we can derive (with $h_j = h_j(P)$)

$$\begin{aligned} (n-j-3)h_j + (j+1)h_{j+1} &\leqslant n \cdot h_j \\ (j+1)h_{j+1} &\leqslant (j+3)h_j \\ h_{j+1} &\leqslant \frac{j+3}{j+1} h_j \ . \end{aligned}$$

This bound can be iterated until we reach h_0:

$$h_{j+1} \leqslant \tfrac{j+3}{j+1} h_j \leqslant \tfrac{j+3}{j+1} \tfrac{j+2}{j} h_{j-1} \leqslant \underbrace{\frac{j+3}{j+1} \frac{j+2}{j} \cdots \frac{3}{1}}_{=\binom{j+3}{2}} \underbrace{h_0}_{\leqslant 1} \leqslant \binom{j+3}{2} \ .$$

Theorem 11.18. *Let P be a set of n points in general position.*
(i) For all j, $0 \leqslant j \leqslant n-3$,

$$h_j = h_{n-3-j} \quad \textit{and} \quad h_j \leqslant \binom{j+2}{2}$$

and hence $h_j \leqslant \min\{\binom{j+2}{2}, \binom{n-1-j}{2}\}$.

(ii)

$$e_3 \leqslant \begin{cases} 2\binom{n/2+1}{3} = \frac{n(n^2-4)}{24} & \textit{for } n \textit{ even, and} \\ 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2} = \frac{n(n^2-1)}{24} & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. (i) is just a summary of what we have derived so far.

For (ii) we simply plug these bounds into relation (11.7). Suppose first that n is even. Then

$$(h_0, h_1, \ldots, h_{n/2-2}) = (h_{n-3}, h_{n-4}, \ldots, h_{n/2-1})$$

and, therefore,

$$e_3 = \sum_{i=0}^{n-3} h_i = 2\sum_{i=0}^{n/2-2} h_i \leqslant 2\sum_{i=0}^{n/2-2} \binom{i+2}{2} = 2\binom{n/2+1}{3}.$$

Second, if n is odd then

$$(h_0, h_1, \ldots, h_{(n-3)/2}) = (h_{n-3}, h_{n-2}, \ldots, h_{(n-3)/2})$$

with $h_{(n-3)/2}$ appearing in both sequences. Then

$$\begin{aligned} e_3 = \sum_{i=0}^{n-3} h_i &= 2\sum_{i=0}^{(n-3)/2-1} h_i + h_{(n-3)/2} \\ &\leqslant 2\sum_{i=0}^{(n-3)/2-1} \binom{i+2}{2} + \binom{(n+1)/2}{2} \\ &= 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2}. \end{aligned}$$

□

There are sets where all these bounds are tight, simultaneously. We find it more convenient to substantiate this claim after some further considerations.

Exercise 11.19. *Show* $e_3 \leqslant \frac{1}{4}\binom{n}{3} + O(n^2)$. *(That is, asymptotically, at most* $1/4$ *of all triangles embrace the origin.)*

Exercise 11.20. *Try to understand the independence of* $\vec{h}$ *of the actual lifting by observing what happens as you move a single point vertically.*

We have obtained lower and upper bounds in the plane. Before proceeding to better methods for computing the e_k's, we generalize to arbitrary dimension d.

11.3 Embracing k-Sets in d-Space

It has been announced that our methods easily carry over to higher dimensions. So let us do a quick tour of deriving the bounds analogous to Theorem 11.18. The reader should make sure that indeed all arguments can be generalized. It is a good exercise to recapitulate the proofs.

Let us now assume that $P \subseteq \mathbb{R}^d$ is a set of n points in general position with the origin 0, i.e., $0 \notin P$ and no $d+1$ points in $P \cup \{0\}$ lie on a common hyperplane. There is no change in the notion of an embracing k-set and of the vector $\vec{e}$, but let us still repeat:

For $k \in \mathbb{N}_0$, we define $e_k = e_k(P) := \left|\left\{A \in \binom{P}{k} \mid 0 \in \text{conv}(A)\right\}\right|$. We call A with $0 \in \text{conv}(A)$ an *embracing* k*-set*. For $|A| = d+1$, we call A an *embracing simplex*. We will still mostly use Δ for embracing simplices. Observe that $e_0 = e_1 = \cdots = e_d = 0$ and $e_{d+1} = \text{sd}_0(P)$.

We consider a generic vertical lifting from P to $\mathbb{R}^{d+1}$, denoted by $p \mapsto p'$. "Generic" means that no $d+2$ points in P' lie in a common hyperplane.

If $\Delta \subseteq P$ is an embracing simplex (with $d+1$ points), then its lifting Δ' spans (via its affine hull) a hyperplane. We use $\beta_\Delta = \beta_{\Delta'}$ for the number of P' below this hyperplane. We emphasize that β_Δ is dependent on the lifting chosen.

As before, we define the vector $\vec{h}$ with

$$h_i = h_i(P') := \text{ the number of embracing simplices } \Delta \text{ with } \beta_\Delta = i.$$

with the only difference that we now consider embracing simplices rather than triangles.

Checkpoint 11.21. $\sum_{i=0}^{n-(d+1)} h_i = e_{d+1}$.

In the plane, our next lemma was: $0 \in P \Leftrightarrow h_0 = h_{n-3} = 1$, where h_{n-3} counted all embracing triangles Δ with all points except for Δ below. This number is now $n-(d+1)$ and we get:

Lemma 11.22. $0 \in P \Leftrightarrow h_0 = h_{n-(d+1)} = 1$

We can take over the proof we have seen for Lemma 11.6. Essential ingredients were that a line, in particular the x_{d+1}-axis, intersects the convex polytope $\text{conv}(P')$ in an interval. If this interval is nonempty, its endpoints are parts of facets of the polytope. The hyperplane spanned by one of these facets has no point in P' below, the hyperplane of the other facet has no point above (here we use the fact that the intersecting line, the x_{d+1}-axis, is vertical). Hence, these facets are liftings of embracing simplices Δ_0 and Δ_1, resp., with $\beta_{\Delta_0} = 0$ and $\beta_{\Delta_1} = n-(d+1)$. Via the notion of a witness embracing simplex $\Delta \subseteq A$ of an embracing k-set A, the counterpart of (11.7) reads now (note $|\Delta| = d+1$)

$$e_k = \sum_{\Delta \text{ embracing}} \binom{\beta_\Delta}{k-(d+1)} = \sum_{i=0}^{n-(d+1)} \binom{i}{k-(d+1)} h_i \, . \tag{11.23}$$

and

$$\vec{e}_{d+1..n} \underset{\text{each other}}{\overset{\text{determine}}{\longleftrightarrow}} \vec{h}_{0..n-(d+1)} \ .$$

Hence, h_i is independent of the lifting chosen and we can write $h_i = h_i(P)$. Symmetry of $\vec{h}$ follows readily, as before, by looking at points above instead of below a lifted embracing simplex.

Lemma 11.24. $h_i = h_{n-(d+1)-i}$.

The two lemmas towards the upper bound carry over, with the first identical to what we have seen before, and the second with the constants adapted to the dimension.

Lemma 11.25. *For all* $j \in \mathbb{N}_0$ *and all* $q \in P$, $h_j(P \setminus \{q\}) \leqslant h_j(P)$.

Lemma 11.26. *For all* $j \in \mathbb{N}_0$, $\sum_{q\in P} h_j(P \setminus \{q\}) = (n-j-(d+1))h_j(P) + (j+1)h_{j+1}(P)$.

Proof. A contribution to $\sum_{q\in P} h_j(P \setminus \{q\})$ can come only from simplices Δ with $\beta_\Delta = j$ or $\beta_\Delta = j+1$ (relative to the complete set P and a chosen lifting P′).

- If $\beta_\Delta = j$, Δ' remains a simplex with j points below, if q is chosen as one of the $(n-(d+1)-j)$ points above.
- If $\beta_\Delta = j+1$, Δ' turns into a simplex with j points below, if q is chosen as one of the $(j+1)$ points below.

□

Again, for the upper bound on the h_i's, just like in the plane, we start with

$$\sum_{q\in P} h_j(P \setminus \{q\}) \leqslant n \cdot h_j(P) \ ,$$

and then continue

$$\begin{aligned}
(n-j-(d+1))h_j + (j+1)h_{j+1} &\leqslant n \cdot h_j \\
(j+1)h_{j+1} &\leqslant (j+d+1)h_j \\
h_{j+1} &\leqslant \frac{j+d+1}{j+1} h_j \ .
\end{aligned}$$

Then we iterate this until we reach h_0:

$$h_{j+1} \leqslant \tfrac{j+d+1}{j+1} h_j \leqslant \tfrac{j+d+1}{j+1}\,\tfrac{j+d}{j} h_{j-1} \leqslant \underbrace{\frac{j+d+1}{j+1}\,\frac{j+d}{j} \cdots \frac{d+1}{1}}_{=\binom{j+d+1}{d}} \underbrace{h_0}_{\leqslant 1} \leqslant \binom{j+d+1}{d} \ .$$

Theorem 11.27. *Let* $P \subseteq \mathbb{R}^d$ *be a set of* n *points in general position.*
(i) For all j*,* $0 \leqslant j \leqslant n-(d+1)$*,*

$$h_j = h_{n-(d+1)-j} \quad \text{and} \quad h_j \leqslant \binom{j+d}{d}$$

and hence $h_j \leqslant \min\{\binom{j+d}{d}, \binom{n-1-j}{d}\}$.
(ii)

$$e_{d+1} \leqslant \begin{cases} 2\binom{(n+d)/2}{d+1} & \text{for } n-d \text{ even, and} \\ 2\binom{(n+d-1)/2}{d+1} + \binom{(n+d-1)/2}{d} & \text{for } n-d \text{ odd.} \end{cases}$$

Proof. (i) is a summary of what we have derived.
For (ii) we simply plug these bounds into relation (11.23). For $n-d$ even we have

$$(h_0, h_1, \ldots, h_{(n-d)/2-1}) = (h_{n-d-1}, h_{n-d-2}, \ldots, h_{(n-d)/2})$$

and, therefore,

$$e_{d+1} = \sum_{i=0}^{n-(d+1)} h_i = 2\sum_{i=0}^{(n-d)/2-1} h_i \leqslant 2\sum_{i=0}^{(n-d)/2-1} \binom{i+d}{d} = 2\binom{(n+d)/2}{d+1}.$$

If $n-d$ is odd then

$$(h_0, h_1, \ldots, h_{(n-(d+1))/2}) = (h_{n-3}, h_{n-2}, \ldots, h_{(n-(d+1))/2})$$

with $h_{(n-(d+1))/2}$ appearing in both sequences. Then

$$\begin{aligned} e_{d+1} = \sum_{i=0}^{n-(d+1)} h_i &= 2\sum_{i=0}^{(n-(d+1))/2-1} h_i + h_{(n-(d+1))/2} \\ &\leqslant 2\sum_{i=0}^{(n-(d+1))/2-1} \binom{i+d}{d} + \binom{(n+d-1)/2}{2} \\ &= 2\binom{(n+d-1)/2}{d+1} + \binom{(n+d-1)/2}{d}. \end{aligned}$$

□

11.4 Embracing Sets vs. Faces of Polytopes

This section exhibits a duality between points sets, such that, roughly speaking, if P is dual to Q, then there is a bijection between the faces of $\mathrm{conv}(P)$ and the embracing sets of Q. This duality is different from the point-line duality you have seen in a previous chapter. Just to demonstrate this, we will have $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^{n-d-1}$ for $n := |P| = |Q|$ and some d, $0 \leqslant d < n$.

In order to describe this duality, and then handling it, we need some linear algebra terminology handy, together with its relation to our targeted notions "embracing" and "supporting hyperplane" (for faces of a polytope). We will approach this smoothly, and I apologize to those who have these matters on top of their head anyway.[4]

11.4.1 Warm-up

Point sequences and matrices. For integers $d, n \in \mathbb{N}_0$, consider a matrix $A \in \mathbb{R}^{n\times d}$. The sequence $S_A := (p_1, p_2, \ldots, p_n)$ of row vectors of A is a sequence of points in $\mathbb{R}^d$ (or we say in $\mathbb{R}^{1\times d}$, if we want to emphasize that these are row vectors). Clearly, vice versa, every sequence of n points in $\mathbb{R}^d$ can be thought of as a matrix in $A \in \mathbb{R}^{n\times d}$. Let us say right away that we abandon here the general position assumption, at least for the time being. In particular, we allow repetitions in a sequence of points.

We will use $\vec{1}$ and $\vec{0}$ for the row or column vector of all 1's and 0's, resp., of the appropriate dimension (i.e., we use also $\vec{0}$ for the origin 0 in the ambient space). Given a vector $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ (row or column), we write $u \geqslant 0$ if $u_i \geqslant 0$ for all $i = 1, \ldots, m$.

Linear and convex combinations. A linear combination $\lambda_1 p_1 + \cdots \lambda_n p_n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{1\times n}$, of the rows of $A \in \mathbb{R}^{n\times d}$ can be compactly written as $\lambda \cdot A \in \mathbb{R}^{1\times d}$. Here are a few simple observations.

1. $\frac{1}{n}\left(\vec{1}\cdot A\right)$ is the centroid[5] of S_A.
2. $\vec{1}\cdot A = \vec{0}$ iff $\vec{0}$ is the centroid of S_A (obviously, if $\vec{1}\cdot A = \vec{0}$ then $\frac{1}{n}\left(\vec{1}\cdot A\right) = \vec{0}$). Another way of reading $\vec{1}\cdot A = \vec{0}$ is that $\vec{1}$ (as column vector) is orthogonal to all column vectors of A.
3. $\lambda \cdot A$, with $\lambda \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$, is a convex combination of S_A.
4. If $\lambda \cdot A = \vec{0}$ with $\vec{0} \neq \lambda \geqslant 0$, then $\vec{0} \in \mathrm{conv}(S_A)$ (if the presumption holds, we can always scale λ to $\lambda' = c\lambda$ with $\sum_{i=1}^n \lambda'_i = 1$ and $\lambda' \cdot A = \vec{0}$).

Just like $\lambda\cdot A$ denotes a linear combination of the rows of A, the product $A\cdot u \in \mathbb{R}^{n\times 1}$, for some $u \in \mathbb{R}^{d\times 1}$, denotes a linear combination of the columns of A. We recall that A has rank d (i.e. the columns are independent) iff there is no $\vec{0} \neq u \in \mathbb{R}^{d\times 1}$ with $A\cdot u = \vec{0}$.

(Supporting) hyperplanes. An oriented hyperplane in $\mathbb{R}^d$ is defined by a vector $v \in R^{d+1}$ (as a column vector):

$$H_v := \{x \in \mathbb{R}^d \mid (x, -1)\cdot v = \sum_{i=1}^{d} v_i x_i - v_{d+1} = 0\} \text{ for } v = \begin{pmatrix} v_1 \\ \vdots \\ v_{d+1} \end{pmatrix} \in \mathbb{R}^{(d+1)\times 1}.$$

[4] Think of it as a warm-up of your linear-algebra-muscles.
[5] Center of gravity, the average of the points.

(Here $(x, -1)$ is the vector x extended by one extra dimension with entry -1.) H_v^+ is the closed positive halfspace of points x with $(x, -1) \cdot v \geqslant 0$. Recall that an oriented hyperplane H_v is supporting for a ploytope $\mathcal{P}$ if $\mathcal{P} \subseteq H_v^+$.

1. The vector $\sigma := (A, -\vec{1}) \cdot v \in \mathbb{R}^{n \times 1}$ indicates the positions of the points p_i relative to H_v: $p_i \in H_v \Leftrightarrow \sigma_i = 0$ and $p_i \in H_v^+ \Leftrightarrow \sigma_i \geqslant 0$. ($(A, -\vec{1})$ denotes the matrix in $\mathbb{R}^{n \times (d+1)}$ obtained from A by extending it by an extra column of -1's).

2. $(A, -\vec{1}) \cdot v \geqslant 0$ iff H_v is a supporting hyperplane of $\text{conv}(S_A)$.

3. If $v_{d+1} = 0$, $v \neq 0$, and $A \cdot v_{1..d} = \vec{0}$, then H_v is a hyperplane through the origin 0 containing all points in S_A. Or, in other words, the column vectors in A are not independent (and the rank of A is strictly less than d), since we can write $\vec{0}$ as a nontrivial linear combination of the column vectors of A.

11.4.2 Orthogonal dual (Gale Duality)

We are now ready to describe a duality between sequences of n points in $\mathbb{R}^d$ and $\mathbb{R}^{n-d-1}$ that is closely related to the Gale transform.

For $d, n \in \mathbb{N}_0$, we call a matrix $A \in \mathbb{R}^{n \times d}$ *legal* if $0 \leqslant d < n$, $\vec{1} \cdot A = \vec{0}$, and if A has full rank d. The conditions for 'legal' translate to the facts that the origin is the center of gravity of the points in S_A, and that there is no hyperplane in $\mathbb{R}^d$ containing all points in S_A, i.e., $\text{conv}(S_A)$ is a full dimensional polytope – an assumption much weaker than general position!

Given legal matrices $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{n \times (n-d-1)}$, we call B *an orthogonal dual of* A, in symbols $A \perp B$, if $A^\top \cdot B = 0^{d \times (n-d-1)}$. In other words, all columns of A are orthogonal to all columns of B. That is, the columns of A span a linear vector space of dimension d orthogonal to the linear space of dimension $n - d - 1$ spanned by the columns of B, and both spaces are orthogonal to $\vec{1}$ (by the legality condition). Hence, given a legal matrix A, there is always an orthogonal dual B which is unique up to linear transformations. Clearly, $A \perp B \iff B \perp A$.[6]

See Figures 11.1 and 11.2 for examples of orthogonal duals and their point sequences.

Lemma 11.28 (Gale Duality). *For* $d, n \in \mathbb{N}_0$, *let* $\mathbb{R}^{n \times d} \ni A \perp B \in \mathbb{R}^{n \times (n-d-1)}$ *be legal matrices with* $A \perp B$, *and let* $(p_i)_{i=1}^n := S_A$ *and* $(p_i^*)_{i=1}^n := S_B$. *For some given* $I \subseteq \{1, 2, \ldots, n\}$, *consider* $F := \{p_i \mid i \in I\}$ *and* $\overline{F^*} := \{p_i^* \mid i \notin I\}$.[7]
(i) *If* F *is contained in a supporting hyperplane of* $\text{conv}(S_A)$ *then* $0 \in \text{conv}(\overline{F^*})$.
(ii) *If* $0 \in \text{conv}(F)$, *then* $\overline{F^*}$ *is contained in a supporting hyperplane of* $\text{conv}(S_B)$.

[6]This convenient symmetry, enforced by the condition $\vec{1} \cdot A^\top = \vec{0}$, is the only difference to the standard Gale transform – apart from expository details.

[7]Note, it's "$\in$" in the definition of F and it's "$\notin$" for $\overline{F^*}$.

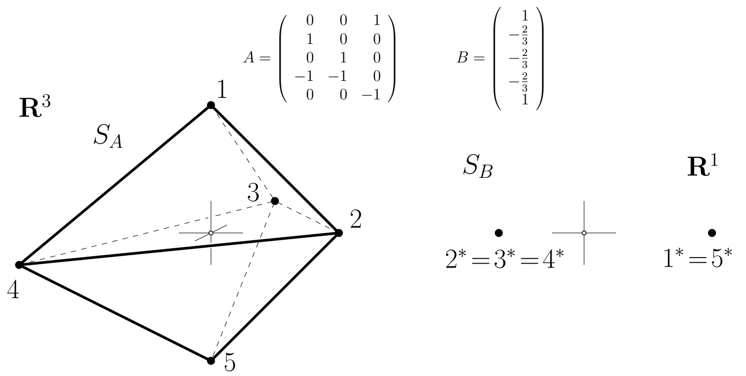

Figure 11.1: *Point sequences* S_A *and* S_B *from orthogonal duals* $\mathbb{R}^{5\times 3} \ni A \perp B \in \mathbb{R}^{5\times 1}$.

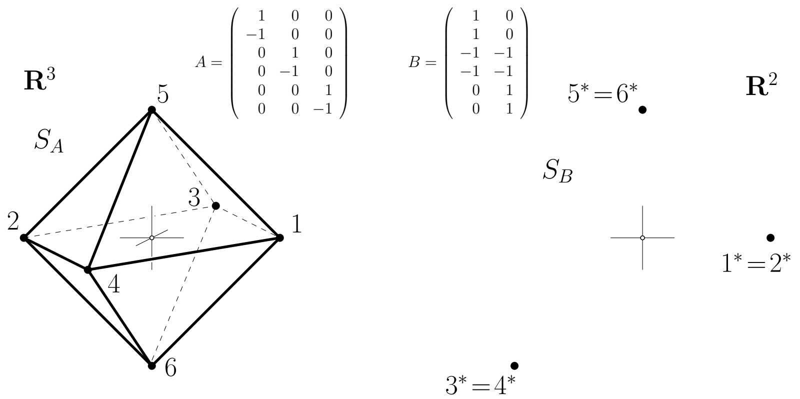

Figure 11.2: *Point sequences* S_A *and* S_B *from orthogonal duals* $\mathbb{R}^{6\times 3} \ni A \perp B \in \mathbb{R}^{6\times 2}$.

Proof. Let F lie in a supporting hyperplane. That is, there is a vector $\nu \in \mathbb{R}^{d+1}$, $\nu_{1..d} \neq \vec{0}$, such that for the vector $\sigma := (A, -\vec{1}) \cdot \nu$ we have $\sigma \geqslant 0$ and $\sigma_i = 0$ for $i \in I$. Note also $\sigma \neq \vec{0}$, since A has full rank. Moreover,

$$\sigma^\top \cdot B = \nu^\top \cdot \underbrace{(A, -\vec{1})^\top \cdot B}_{0^{(d+1)\times(n-d-1)}} = \vec{0} \in \mathbb{R}^{n-d-1}$$

which means that the origin in $\mathbb{R}^{n-d-1}$ is a nontrivial nonnegative linear (and thus convex) combination of points p_i^* with $i \notin I$. That is, $0 \in \text{conv}(\overline{F^*})$.

For the reverse direction (ii), let $\lambda \in \mathbb{R}^{1\times n}$ be a vector that witnesses the fact that $\vec{0} \in \text{conv}(F)$. That is, $\vec{0} \neq \lambda \geqslant 0$, $\lambda \cdot A = \vec{0}$, and $\lambda_i = 0$ for $i \notin I$. $\lambda^\top$ is orthogonal to the linear space spanned by the columns in A (that's exactly what $\lambda \cdot A = \vec{0}$ says). Consequently, it is in the linear space spanned by the columns of $(B, -\vec{1})$, and there is a vector $\nu \in R^{(n-d)\times 1}$ with $(B, -\vec{1}) \cdot \nu = \lambda^\top$. Hence, ν corresponds to a supporting hyperplane that contains all p_i^* with $\lambda_i = 0$. Since $\lambda_i = 0$ for $i \notin I$, the hyperplane contains all points in $\overline{F^*}$. □

Faces in simplicial polytopes. At this point recall (and you probably already did) the discussion about simplicial polytopes, i.e., polytopes where every face is a simplex. We have argued that simplicial polytopes maximize the number of facets, and actually of all i-faces, $0 \leqslant i \leqslant d-1$, for a given number n of vertices.

We also saw that in such a simplicial polytope, every i-face, $0 \leqslant i \leqslant d-1$, has $i+1$ vertices, which determine the face. Also, whenever F is a set of vertices of a face, then the convex hull of every subset is a face of the polytope. F is the vertex set of an i-face iff $|F| = i+1$ and F lies in a supporting hyperplane.[8]

Given a d-dimensional polytope $\mathcal{P}$, let

$$f = f(\mathcal{P}) = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{N}_0^{d+1}$$

be the vector with f_i the number of i-faces, we call this the f-*vector* of $\mathcal{P}$. (Recall that there is the empty face, which we agree to be -1-dimensional; in this vector we ignore the d-dimensional face, the whole polytope itself). For example, $f_{-1} = 1$, f_0 is the number of vertices of $\mathcal{P}$ and f_{d-1} is the number of facets of $\mathcal{P}$.

Observation 11.29. *If $\mathcal{P}$ is a d-dimensional simplicial polytope with vertex set $V(\mathcal{P})$, then f_i is the number of $(i+1)$-element subsets of $V(\mathcal{P})$ that are contained in a supporting hyperplane of $\mathcal{P}$.*

We are ready to employ Gale Duality.

Lemma 11.30. *Let $P \subseteq \mathbb{R}^d$ be a set of n points in general position and let $\mathcal{P} := \mathrm{conv}(P)$, a simplicial polytope, with f-vector $f = (f_{-1}, f_0, \ldots, f_{d-1})$. Suppose P is the set of points in S_A for a legal matrix $A \in \mathbb{R}^{n \times d}$. Moreover, suppose B, a legal matrix, is an orthogonal dual of A such that the point set P^* of points in S_B is a set of n points in general position with the origin 0 in $\mathbb{R}^{d^*}$, $d^* := n-d-1$. Then*

$$f_{i-1} = e_{n-i}(P^*) \ .$$

In particular, $f_{-1} = e_n = 1$, the number of vertices is $f_0 = e_{n-1}$, and the number of facets $f_{d-1} = e_{n-d} = e_{d^*+1}$ (the simplicial depth of 0 in P^*).

Proof. Since $\mathcal{P}$ is simplicial, the number of $(i-1)$-faces is exactly the number of i-element subsets of P that are contained in a supporting hyperplane (Observation 11.29). $\{p_j\}_{j \in J}$, $|J| = i$, is in a supporting hyperplane iff $\{p_j^*\}_{j \notin J}$ is embracing, a subset of $n-i$ elements of P^* (Gale Duality Lemma 11.28). □

In order to apply the upper bounds for the e^k's for the f_i's, we still have to ensure that for every polytope, or at least for every face lattice, the presumptions of the lemma can be achieved. So we start with a simplicial polytope $\mathcal{P}$. We translate and perturb

[8]For all of this it is important that the polytope is simplicial. Think of a cube in 3 dimensions: The facets, which are 2-faces, have vertex sets of size 4, and 3-element subsets of these 4-element sets are *not* vertex sets of any face.

the vertex set $V(\mathcal{P})$ of $\mathcal{P}$ to a polytope $\mathcal{P}'$ with $f(\mathcal{P}') = f(\mathcal{P})$ where 0 is the centroid and the vertex set $V(\mathcal{P}')$ is in general position together with 0. That this is possible needs a not too difficult careful argument, which we sweep under the rug here. Moreover, under these assumptions, an orthogonal dual of $V(\mathcal{P}')$ is a set that is in general position even with the origin (see Exercise 11.34).

Here, finally, comes the bound on the number of faces, first shown by McMullen in 1970.

Theorem 11.31 (Upper Bound Theorem). *Let $\mathcal{P}$ be a simplicial d-dimensional polytope with n vertices and f-vector $f = (f_{-1}, f_0, \ldots, f_{d-1})$. Then there is a vector $h = (h_0, h_1, \ldots, h_d) \in \mathbb{N}_0^{d+1}$ with*

$$f_{i-1} = \sum_{j=0}^{d} \binom{d-j}{i-j} h_j \text{ , and for all } j, \quad h_j \leqslant \binom{j+n-d-1}{j} \quad \text{and} \quad h_j = h_{d-j} \text{ .}$$

In particular,

$$f_{d-1} \leqslant \left\{ \begin{array}{ll} 2\binom{n-\frac{d+1}{2}}{\frac{d-1}{2}} & d \text{ odd} \\ 2\binom{n-\frac{d}{2}-1}{\frac{d}{2}-1} + \binom{n-\frac{d}{2}-1}{\frac{d}{2}} & d \text{ even} \end{array} \right\} = O(n^{\lfloor d/2 \rfloor})$$

The proof of the theorem is just a transformation of Theorem 11.27 via Lemma 11.30. Let us first check the bounds for f_{d-1} for the values we are familiar with: For $d = 2$ we get $f_1 \leqslant 2\binom{n-2}{0} + \binom{n-2}{1} = 2 + n - 2 = n$, and for $d = 3$ we get $f_2 \leqslant 2\binom{n-2}{1} = 2n - 4$, which are both the values to be expected. For $d = 4$, we see already the bound grows beyond linear: $f_3 = 2\binom{n-3}{1} + \binom{n-3}{2} = \frac{n(n-3)}{2}$.

Proof. Let $A \perp B$ with S_A an ordering of the vertices of $V(\mathcal{P}')$, with $\mathcal{P}'$ a perturbation of $\mathcal{P}$ as described above (with $f(\mathcal{P}') = f(\mathcal{P})$). We know that there is a vector $h = (h_0, h_1, \ldots, h_{n-(d^*+1)})$ such that

$$e_k(S_B) = \sum_{j=0}^{n-(d^*+1)} \binom{j}{k-(d^*+1)} h_j \quad \text{and} \quad h_j = h_{n-(d^*+1)-j} \text{ .}$$

Now (with (i) $n - (d^* + 1) = d$, (ii) the symmetry of $h_j = h_{d-j}$, and (iii) $\binom{a}{b} = \binom{a}{a-b}$)

$$\begin{aligned} f_{i-1} = e_{n-i} &= \sum_{j=0}^{n-(d^*+1)} \binom{j}{n-i-(d^*+1)} h_j \\ &\overset{(i)}{=} \sum_{j=0}^{d} \binom{j}{d-i} h_j \overset{(ii)}{=} \sum_{j=0}^{d} \binom{j}{d-i} h_{d-j} = \sum_{j=0}^{d} \binom{d-j}{d-i} h_j \overset{(iii)}{=} \sum_{j=0}^{d} \binom{d-j}{i-j} h_j \end{aligned}$$

Moreover, we have

$$h_j \leqslant \binom{j+d^*}{d^*} = \binom{j+n-d-1}{n-d-1} = \binom{j+n-d-1}{j}$$

Finally, (with $\frac{n+d^*}{2} = n - \frac{d+1}{2}$)

$$\begin{aligned} f_{d-1} &= e_{n-d} = e_{d^*+1} \leqslant \left\{ \begin{array}{ll} 2\binom{(n+d^*)/2}{d^*+1} & n-d^* \text{ even} \\ 2\binom{(n+d^*-1)/2}{d^*+1} + \binom{(n+d^*-1)/2}{d^*} & n-d^* \text{ odd} \end{array} \right\} \\ &= \left\{ \begin{array}{ll} 2\binom{n-\frac{d+1}{2}}{\frac{d-1}{2}} & d \text{ odd} \\ 2\binom{n-\frac{d}{2}-1}{\frac{d}{2}-1} + \binom{n-\frac{d}{2}-1}{\frac{d}{2}} & d \text{ even} \end{array} \right\} = O(n^{\lfloor d/2 \rfloor}) \end{aligned}$$

□

Tightness. The bounds in the Upper Bound Theorem are tight, for the whole f-vector. Polytopes that attain this bound are quite easy to describe: So-called *cyclic polytopes* which are obtained as the convex hull of set $\{(t, t^2, \ldots, t^d) \mid t = 1, 2, \ldots, n\}$. We skip the proof here, that shows that such polytopes have indeed that many faces of various dimension. These polytopes have the property that for all $i \leqslant \lfloor \frac{d}{2} \rfloor$, all i-element subsets of the vertices form faces, $(i-1)$-faces. For example, in 4-space, all pairs of vertices are connected by an edge of the cyclic polytope.

The beauty of the theorem goes much beyond supplying an upper bound. Many facts known about polytopes follow now quite naturally.

Identities (Dehn-Sommerville relations). The identities $h_j = h_{d-j}$, $j = 0, 1, \ldots, d$, are called the *Dehn-Sommerville relations*, some special cases are known to you already. To see that, recall that the h-vector is unique and there is a formula

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1} .$$

We get $h_0 = (-1)^0 \binom{d}{d} f_{-1} = 1$ (which nicely fits with what we expected) and

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} - \cdots (-1)^{d-1} f_0 + (-1)^d$$

which, by $h_d = h_0$, has to be 1. For $d = 2$ this relation reads $f_1 - f_0 = 0$ (matches our expectation that a convex polygon has the same number of edges as vertices) and for $d = 3$ we get $f_2 - f_1 + f_0 = 2$, which is the Euler Relation for convex polytopes in $\mathbb{R}^3$. The general identity is called Euler-Poincaré Formula.

More formulas of the type are $2f_{d-2} = df_{d-1}$, which can be easily obtained directly by double-counting.

The usual proof. The "usual proof" of the Upper Bound Theorem does not take the detour to the Gale dual. Instead, the h-vector is defined directly for a simplicial convex polytope $\mathcal{P} \subseteq \mathbb{R}^d$. The ingredients of the proof are similar, actually the same as we saw translated to the Gale Dual. Apart from the original paper by McMullen, see, e.g., the book on polytopes by Günter Ziegler for this version of the proof.

Exercise 11.32. *Let* $A \in \mathbb{R}^{n\times d}$ *and* $B \in \mathbb{R}^{n\times(n-d-1)}$ *be legal matrices with* $A \perp B$*, such that* S_A *and* S_B *are in general position with the origin (in particular, all points are distinct).*

(i) Suppose that all elements in S_A *are extreme (i.e. vertices of the* $\text{conv}(S_A)$*. What does this translate to for the embracing sets of* S_B*?*

(ii) Suppose $f_{i-1}(\text{conv}(S_A)) = \binom{n}{i}$*. What does this translate to for the embracing sets of* S_B*?*

Exercise 11.33. *Show that a simplicial* d*-dimensional polytope* $\mathcal{P}$ *with* $d+2$ *vertices has always* $k(d+2-k)$ *facets, for some* k*,* $2 \leqslant k \leqslant d$*.*

Exercise 11.34. *Let* $\mathbb{R}^{n\times d} \ni A \perp B \in \mathbb{R}^{n\times n-d-1}$ *for legal matrices with* A, B*. Let* $(p_i)_{i=1}^n = S_A$ *and let* $(p_i^*)_{i=1}^n = S_B$ *and let* $I \subseteq \{1, 2, \ldots, n\}$ *(there are no general position assumptions other than the legality of* A *and* B*, in particular,* S_A *or* S_B *may contain repeated points).*

(i) Suppose $|I| = d+1$*, and points* $\{p_i\}_{i\in I}$ *lie in a common hyperplane. What does this translate to for* S_B*?*

(ii) Suppose $|I| = d$*, and points* $\{p_i\}_{i\in I}$ *lie in a common hyperplane with the origin* $0 \in \mathbb{R}^d$*. What does this translate to for* S_B*?*

(iii) Show that S_A *is generic iff* S_B *is generic. Here we call a sequence* $(p_i)_{i=1}^n$ *of points in* $\mathbb{R}^d$ generic *if* $|\{p_i\}_{i=1}^n \cup \{\vec{0}\}| = n+1$ *and if no* $d+1$ *points in* $\{p_i\}_{i=1}^n \cup \{\vec{0}\}$ *lie in a common hyperplane.*

Exercise 11.35. *Let* $A \in \mathbb{R}^{n\times d}$ *and* $B \in \mathbb{R}^{n\times n-d-1}$ *legal matrices with* $A \perp B$*, such that* S_A *and* S_B *are in general position with the origin (in particular, all points are distinct). Suppose* n *is even.*

We call a vector $\lambda \in \mathbb{R}^n$ balanced*, if no entry of* λ *is* 0*, and there is the same number of positive and negative entries in* λ*. We call* (Q^+, Q^-) *a feasible equipartition of the set of points in* $S_A = (p_1, p_2, \ldots, p_n)$ *if there is a balanced vector* λ *such that* $\lambda \cdot A = \vec{0}$ *and* $Q^+ = \{p_i \, in \, S_A \mid \lambda_i > 0\}$ *and* $Q^- = \{p_i \, in \, S_A \mid \lambda_i < 0\}$*. What do these feasible equipartitions translate to for the points of* S_B*?*

11.5 Faster Counting in the Plane – Another Vector

Call a directed edge $0q$, $q \in P$, an i-*edge*, if i points in P lie to the left of the directed line through $0q$ (directed from 0 to q). Let $\ell_i = \ell_i(P)$ be the number of i-edges of P.

Checkpoint 11.36. $\sum_i \ell_i = n$*. What is the vector* $\vec{l} = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ *for the case* $0 \notin \text{conv}(P)$*?*

For every nonempty set $A \subseteq P$ with $0 \notin \text{conv}(A)$, $\text{conv}(A)$ has a left and a right tangent from 0. Let $q \in A$ be the touching point of the right tangent. For how many k-element sets $A \subseteq P$ with $0 \notin \text{conv}(A)$ is this point q the right touching point?

Checkpoint 11.37. *This is* $\binom{i}{k-1}$ *if* $0q$ *is an* i*-edge.*

Hence, we have for $1 \leqslant k \leqslant n$:

$$e_k = \underbrace{\binom{n}{k}}_{\sum_{i=0}^{n-1}\binom{i}{k-1}} - \sum_{i=0}^{n-1}\binom{i}{k-1}\ell_i = \sum_{i=0}^{n-1}\binom{i}{k-1}(1-\ell_i) \ . \tag{11.38}$$

We have a combinatorial interpretation of the z_i's in (11.4) and, therefore, numbers ℓ_i satisfying (11.38) are unique.

Exercise 11.39. *Show that* $\ell_i = \ell_{n-1-i}$*. (Hint: Wonder why we chose "left" and not "right".)*

We can compute the vector $\vec{l}_{0..n-1}$ in $O(n \log n)$ time. For that we rotate a directed line about 0, starting with the horizontal line, say. We always maintain the number of points left of this line, and update this number whenever we sweep over a point $q \in P$. This q may lie ahead of 0 or behind it; depending on this the number increases by 1 or decreases by 1, resp. In this way, with a rotation by 180 degrees, we can compute the "number of points to the left" for every $q \in P$. We need $O(n \log n)$ time to sort the events (encounters with points in P). We initialize the "number to the left" in $O(n)$ time in the beginning, and then update the number in $O(1)$ at each event. This gives $O(n \log n)$ altogether.

Theorem 11.40. *In the plane, the simplicial depth* $\text{sd}_q(P)$ *can be computed in* $O(n \log n)$ *time, provided* $P \,\dot\cup\, \{q\}$ *is in general position.*

Clearly, all entries e_k, $1 \leqslant k \leqslant n$, can be computed based on the vector $\vec{l}$. However, keep in mind that the binomial coefficients involved in the sum (11.38) must be determined and that the numbers involved are large (up to n-bit numbers).

Showing that the upper bound in Theorem 11.18 is tight is now actually easy.

If P is the set of vertices of a regular n-gon, n odd, centered at 0, then $\ell_{(n-1)/2} = n$ (and all other ℓ_i's vanish). Therefore,

$$e_3 = \binom{n}{3} - \binom{(n-1)/2}{2} \cdot n = \frac{n(n^2-1)}{24} \ ,$$

and the case of n odd is shown tight in Theorem 11.18.

For n even, consider the vertices of a regular n-gon centered at 0, and let P be a slightly perturbed set of these vertices so that $P \cup \{0\}$ is in general position. Note that all edges $0q$, $q \in P$, must be $(n/2-1)$- or $(n/2)$-edges. Interestingly, because of the symmetry of the ℓ-vector, we immediately know that $\ell_{n/2-1} = \ell_{n/2} = n/2$ (with all other ℓ_i's vanishing), independent of our perturbation. Now

$$e_3 = \binom{n}{3} - \left(\binom{n/2-1}{2} + \binom{n/2}{2}\right)\frac{n}{2} = \frac{n(n^2-4)}{24},$$

and Theorem 11.18 is proven tight also for n even.

A next step is to understand what the possible ℓ-vectors for n points are, and in this way characterize and eventually count all possibilities for $\vec{l}$ and thus for $\vec{e}$.

11.5.1 Characterizing All Possibilities

We start with two observations about properties of $\vec{l}$.

Exercise 11.41. *Show that $\ell_{\lfloor (n-1)/2 \rfloor} \geqslant 1$. (There is always a halving edge.)*

Exercise 11.42. *Show that if $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for all j, $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$.*

We summarize our knowledge about $\vec{l}$.

Theorem 11.43. *For $n \in \mathbb{N}$, the vector $\vec{l}_{0..n-1}$ of an n-point set satisfies the following conditions.*

- *All entries are nonnegative integers.*
- $\sum_{i=0}^{n-1} \ell_i = n$.
- $\ell_i = \ell_{n-1-i}$. *(Symmetry)*
- *If $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for all j, $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. (remains positive towards the middle)*

Let us call a vector of length n a *legal n-vector* if the conditions of Theorem 11.43 are satisfied. Then (1) is the only legal 1-vector, $(1,1)$ is the only legal 2-vector, and the only legal 3-vectors are $(0,3,0)$ and $(1,1,1)$. The following scheme displays how we derive legal 6-vectors from legal 5-vectors, and how we can derive legal 7-vectors from legal 5- or 6-vectors.

$$\overbrace{\begin{matrix}0&0&5&0&0\\0&1&3&1&0\\0&2&1&2&0\\1&1&1&1&1\end{matrix}}^{n=5} \quad \xrightarrow{\text{add 1 in the middle and split}} \quad \overbrace{\begin{matrix}0&0&3&3&0&0\\0&1&2&2&1&0\\0&2&1&1&2&0\\1&1&1&1&1&1\end{matrix}}^{n=6}$$

$$\text{add 2 in the middle} \downarrow \qquad\qquad \text{insert a 1 in the middle} \downarrow$$

$$\underbrace{\begin{matrix}0&0&0&7&0&0&0\\0&0&1&5&1&0&0\\0&0&2&3&2&0&0\\0&1&1&3&1&1&0\end{matrix}}_{n=7,\text{ with }>1\text{ in the middle}} \qquad \underbrace{\begin{matrix}0&0&3&1&3&0&0\\0&1&2&1&2&1&0\\0&2&1&1&1&2&0\\1&1&1&1&1&1&1\end{matrix}}_{n=7,\text{ with }=1\text{ in the middle}}$$

Exercise 11.44. *Show that the scheme described, when applied to general n odd, is complete. That is, starting with all legal n-vectors, n odd, we get all legal $(n+1)$-vectors, and from the n- and $(n+1)$-vectors, we get all $(n+2)$-vectors.*

Exercise 11.45. *Show that the number of legal n-vectors is exactly $2^{\lfloor (n-1)/2 \rfloor}$.*

Exercise 11.46. *Show that every legal n-vector is the ℓ-vector of some set of n points in general position.*

With these exercises settled, we have given a complete characterization of all possible ℓ-vectors, thus of all possible e-vectors.

Theorem 11.47. *The number of different e-vectors (or ℓ-vectors) for n points is exactly $2^{\lfloor (n-1)/2 \rfloor}$.*

Exercise 11.48. *Show that $\sum_{i=0}^{j} \ell_i \leqslant j+1$ for all $0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. (Hint: Otherwise, we get into conflict with "remains positive towards the middle").*

11.5.2 Some Add-Ons

We are still missing an interpretation of the y_i's in relations (11.3). We want to leave this as an exercise.

Exercise 11.49. *For a set P of n points in general position, consider the vector*

$$(b_0, b_1, \ldots, b_{n-2})$$

defined by the relations

$$e_k = \binom{n}{k} - \sum_{i=0}^{n-2} \binom{i}{k-2} b_i = \sum_{i=0}^{n-2} \binom{i}{k-2} (n - i - 1 - b_i) \ ,$$

for $2 \leqslant k \leqslant n$. Give a combinatorial interpretation of these numbers b_i, $0 \leqslant i \leqslant n-2$.

Finally, let us investigate how the vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ from relations (11.2), (11.3), and (11.4) connect to each other. Clearly, e_1 and e_2 given, they determine each other. But how? This will allow us to relate the vectors $\vec{h}$ and $\vec{l}$.

Exercise 11.50. *Consider the relations defined in the beginning of this section on $\vec{x}_{0..n-3}, \vec{y}_{0..n-2}$, $\vec{z}_{0..n-1}$, and $\vec{e}_{1..n}$ (using $e_1 = e_2 = 0$). Then the y_i's are the forward differences of the x_i, and the z_i's are the forward differences of the y_i's. Prove this. More concretely, show that*

$$y_i = \begin{cases} -x_0 & i = 0 \\ x_{i-1} - x_i & 1 \leqslant i \leqslant n-3 \\ x_{n-3} & i = n-2 \end{cases}$$

or, equivalently,

$$y_i = x_{i-1} - x_i \quad \textit{for all } 0 \leqslant i \leqslant n-2, \textit{ with } x_{-1} := x_{n-2} := 0.$$

Show that this entails as well

$$x_i = -\sum_{j=0}^{i} y_j \quad , \textit{for } 0 \leqslant i \leqslant n-3.$$

Exercise 11.51. *Prove for vectors $\vec{a}_{0..m}$ and $\vec{b}_{0..m}$*

$$\forall k, 0 \leqslant k \leqslant m: \quad a_k = \sum_{i=0}^{m} \binom{i}{k} b_i$$

$$\iff \forall i, 0 \leqslant i \leqslant m: \quad b_i = \sum_{k=0}^{m} (-1)^{i+k} \binom{k}{i} a_k \ .$$

Exercise 11.52. *Employing the previous exercise, what does $h_0 = 1$ say about $\vec{e}_{3..n}$.*

The following facts can now be readily derived.

Theorem 11.53.

$$h_i = \binom{i+2}{2} - \sum_{j=0}^{i} (i+1-j)\ell_j$$

Exercise 11.54. *Prove Theorem 11.53.*

Note that this implies the upper bounds we proved for the h_i's in Theorem 11.18, since $\sum_{j=0}^{i}(i+1-j)\ell_j$ is always nonnegative. Moreover, a combinatorial interpretation of the slack becomes evident.

Theorem 11.55.

$$e_k = \sum_{i=0}^{n} \binom{i}{k} (\ell_i - \ell_{i-1}) \quad \text{with } \ell_{-1} = \ell_n = 1$$

Exercise 11.56. *Prove Theorem 11.55.*

Let us point out other counting problems which can be solved efficiently with the insights developed.

Exercise 11.57. *Given a ray r (emanating from point q) and n points P in the plane, design an efficient algorithm that counts the number of points connecting segments intersecting r. You may assume that $P \cup \{q\}$ is in general position and that r is disjoint from P.*

Exercise 11.58. *Let w be a line minus an interval on it (an infinite wall with a window). Given n points P in the plane, design an efficient algorithm that counts the number of point connecting segments disjoint from w (i.e. the number of pairs of points that see each other, either because they are both on the same side of w or because they see each other through the window in w. You may assume general position.*

Exercise 11.59. *Recall that a point c is a centerpoint of P if every halfplane containing c contains at least $|P|/3$ points in P.*

Identify the properties of $\vec{e}$, $\vec{h}$ and $\vec{l}$ that show that 0 is a centerpoint of P.

Exercise 11.60. *Show that $y_i = -y_{n-2-i}$ and $y_i \leqslant 0$ for all $0 \leqslant i \leqslant \lfloor \frac{n-2}{2} \rfloor$. (We refer here to the y_i's as defined by relations (11.3) at the beginning of this section. Hint: You may wish to recall Homework 11.50 and Exercise 11.48.)*

Exercise 11.61. *Show that $h_i \geqslant h_{i-1}$ for all $0 \leqslant \lfloor \frac{n-3}{2} \rfloor$.*

Questions

65. Explain how the h-vector of a planar point set is defined via a lifting. Give the relation between the e-vector (number of embracing k-sets) and the h-vector.

All of the following five questions also include Question 65.

66. Argue, why the h-vector is independent of the lifting.
67. Argue, why the h-vector is symmetric.
68. Argue, why for a given generic lifting $P' \subseteq \mathbb{R}^3$ of a point set $P \subseteq \mathbb{R}^2$ in general position, removing a point (with its lifting) cannot increase h_j, i.e. $h_j(P \setminus \{p\}) \leqslant h_j(P)$ for all $j \in \mathbb{N}_0$ and all $p \in P$.
69. Show how the ℓ-vector can be computed in $O(n \log n)$ time.
70. Argue why the ℓ-vector is symmetric ($\ell_i = \ell_{n-1-i}$ for all i, $0 \leqslant i \leqslant n-1$).
71. Explain orthogonal duals (Gale Duality). How do embracing sets and faces of polytopes relate to each other?