# Chapter 4

# Crossings

So far within this course we have mostly tried to avoid edge crossings and studied planar graphs, which allow us to avoid crossings altogether. However, without doubt there are many interesting graphs that are not planar, and still we would like to draw them in a reasonable fashion. An obvious quantitative approach is to avoid crossings as much as possible, even if they cannot be avoided completely.

## 4.1 Crossing Numbers

For a graph $G = (V, E)$, the *crossing number* $\mathrm{cr}(G)$ is defined as the minimum number of edge crossings over all drawings of $G$. In an analogous fashion, the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ is defined as the minimum number of edge crossings over all straight-line drawings of $G$. A drawing of $G$ with $\mathrm{cr}(G)$ or $\overline{\mathrm{cr}}(G)$ crossings is called a *minimum-crossing drawing* or *minimum-crossing straight-line drawing*, respectively.

In order to see that these notions are well-defined, let us first argue that the number of crossings in a minimum-crossing drawing is finite and, in fact, upper bounded by $\binom{|E|}{2}$. To see this it suffices to consider a straight-line drawing of $G$ on a set $P$ of $n = |V|$ points in general position (no three collinear). Bijectively map the vertices of $G$ to the points of $P$ in an arbitrary manner. Then draw every edge as a straight-line segment. As $P$ is in general position, we obtain a valid drawing, in which every pair of distinct edges has at most one common point. In fact, this latter property holds for *all* minimum-crossing drawings, as the following lemma demonstrates.

**Lemma 4.1.** *In a drawing of a graph $G$ with $\mathrm{cr}(G)$ crossings, every two distinct edges share at most one point.*

*Proof.* Consider any drawing $\Gamma$ of $G$ with $\mathrm{cr}(G)$ crossings, and suppose for a contradiction that there are two distinct edges $e$ and $f$ that share two distinct points $p$ and $q$ in $\Gamma$. Without loss of generality suppose that $e$ has at most as many crossings on its arc between $p$ and $q$ as $f$ has. Then redraw $f$ to closely follow $e$ between $p$ and $q$ and so that $e$ and $f$ do not cross at $p$; see Figure 4.1 for illustration. If $f$ has self-crossings due to this modification, eliminate them (by omitting the arc between the two occurrences of a self-crossing). The resulting drawing $\Gamma'$ has at least one fewer crossing than $\Gamma$, in contradiction to $\Gamma$ having $\mathrm{cr}(G)$ crossings only. □

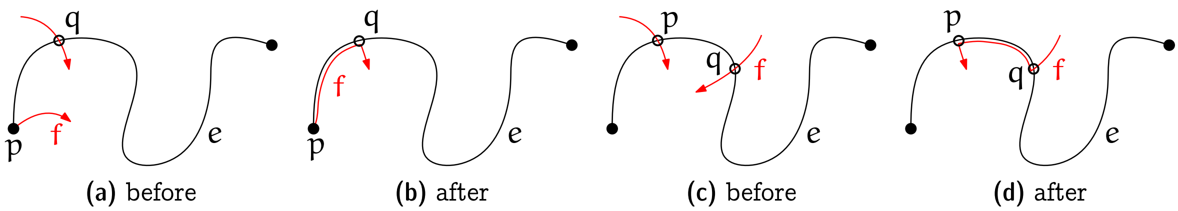


**Figure 4.1:** *Redraw an edge $f$ between two common points with another edge $e$ to reduce the overall number of crossings. Figure (a) and (b) depict the situation where $e$ and $f$ are adjacent at $p$; in this case the crossing at $q$ can be eliminated. Figure (c) and (d) depict the situation where both $p$ and $q$ are crossings; in this case it depends on the local direction of $f$ at these crossings: If $f$ approaches $e$ from different sides at $p$ and $q$, then both crossings can be eliminated. Otherwise, as depicted here, one crossing remains.*

In particular, by Lemma 4.1 no two adjacent edges (that is, edges that have a common endpoint) cross in a minimum-crossing drawing. A drawing in which no two adjacent edges cross is called *star-simple* because for every vertex the star of incident edges forms a plane subdrawing.[1] A drawing in which every pair of edges has at most one point in common is called *simple*, and the corresponding geometrically represented graph is called a *simple topological graph*. So, using this terminology, Lemma 4.1 could also be stated as "Every minimum-crossing drawing is simple."

It is quite easy to give an upper bound on the crossing number of a particular graph, simply by describing a drawing and counting the number of crossings in this drawing. Conversely, it is much harder to give a lower bound on the crossing number of a graph because such a bound corresponds to a statement about *all* possible drawings of this graph. But the following simple lower bound can be obtained by counting edges.

**Lemma 4.2.** *For a graph $G$ with $n \geqslant 3$ vertices and $e$ edges, we have* $\mathrm{cr}(G) \geqslant e-(3n-6)$.

*Proof.* Consider a drawing of $G = (V, E)$ with $\mathrm{cr}(G)$ crossings. For each crossing, pick one of the two involved edges arbitrarily. Obtain a new graph $G' = (V, E')$ from $G$ by removing all picked edges. By construction $G'$ is plane and, therefore, $|E'| \leqslant 3n - 6$ by Corollary 2.5. As at most $\mathrm{cr}(G)$ edges were picked (some edge could be picked for several crossings), we have $|E'| \geqslant |E| - \mathrm{cr}(G)$. Combining both bounds completes the proof. □

[1] In the literature also the terms *semi-simple* or *semisimple* are used.

## 4.2 The Crossing Lemma

The bound in Lemma 4.2 is quite good if the number of edges is close to $3n$ but not so good for dense graphs. For instance, for the complete graph $K_n$ the lemma guarantees a quadratic number of crossings, whereas according to the Guy-Harary-Hill Conjecture [5]

$$\operatorname{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \in \Theta(n^4).$$

The conjecture has been verified, in part with extensive computer help, for the complete graph on $n \leqslant 14$ vertices [2, 6, 8]; but it remains open for $n \geqslant 15$.

So for a dense graph $G$ we should try a different approach. Given that the bound in Lemma 4.2 is not so bad for sparse graphs, why not apply it to some sparse subgraph of $G$? This astonishingly simple idea turns out to work really well, as the following theorem demonstrates.

**Theorem 4.3** (Crossing Lemma [4]). *For a graph $G$ with $n$ vertices and $e \geqslant 4n$ edges, we have $\operatorname{cr}(G) \geqslant e^3/(64n^2)$.*

*Proof.* Consider a drawing of $G$ with $\operatorname{cr}(G)$ crossings. Take a random induced subgraph of $G$ by selecting each vertex independently with probability $p$ (a suitable value for $p$ will be determined later). By this process we obtain a random subset $U \subseteq V$ and the corresponding induced subgraph $G[U]$, along with an induced drawing for $G[U]$. Consider the following three random variables:

- $N$, the number of vertices selected, with $E[N] = pn$;
- $M$, the number of edges induced by the selected vertices, with $E[M] = p^2e$; and
- $C$, the number of crossings induced by the selected vertices and edges, with $E[C] = p^4\operatorname{cr}(G)$. (Here we use that adjacent edges do not cross in a minimum-crossing drawing by Lemma 4.1.)

According to Lemma 4.2, these quantities satisfy $C \geqslant \operatorname{cr}(G[U]) \geqslant M - 3N$. Taking expectations on both sides and using linearity of expectation yields $E[C] \geqslant E[M] - 3E[N]$ and so $p^4\operatorname{cr}(G) \geqslant p^2e - 3pn$. Setting $p = 4n/e$ (which is $\leqslant 1$ due to the assumption $e \geqslant 4n$) gives

$$\operatorname{cr}(G) \geqslant \frac{e}{p^2} - 3\frac{n}{p^3} = \frac{e^3}{16n^2} - 3\frac{e^3}{64n^2} = \frac{e^3}{64n^2}$$

□

Asymptotically the bound in Theorem 4.3 is tight: Pach and Tóth [7] describe graphs with $n \ll e \ll n^2$ that have crossing number at most

$$\frac{16}{27\pi^2}\frac{e^3}{n^2} < \frac{1}{16.65}\frac{e^3}{n^2}.$$

Hence it is not possible to replace $1/64$ by $1/16.65$ in the statement of the theorem. However, the constant $1/64$ is not the best possible: Ackerman [1] showed that $1/64$ can be replaced by $1/29$, at the cost of requiring $e \geqslant 7n$. The beautiful proof described above is also listed in "Proofs from THE BOOK" [3, Chapter 40], where it is attributed to Chazelle, Sharir, and Welzl; the original proof was more complicated and had a worse constant.

## 4.3 Applications of the Crossing Lemma

In the remainder of this chapter, we will discuss several nontrivial bounds on the size of combinatorial structures that can be obtained by a more-or-less straightforward application of the Crossing Lemma. These beautiful connections were observed by Székely [9], the original proofs were different and more involved.

**Theorem 4.4** (Szemerédi-Trotter [10]). *The maximum number of incidences between $n$ points and $m$ lines in $\mathbb{R}^2$ is at most $2^{5/3} \cdot n^{2/3}m^{2/3} + 4n + m$.*

*Proof.* Let $P$ denote the given set of $n$ points, and let $L$ denote the given set of $m$ lines. We may suppose that every line from $L$ contains at least one point from $P$. (Discard all lines that do not; they do not contribute any incidence.) Denote by $I$ the number of incidences between $P$ and $L$. Consider the graph $G = (P, E)$ whose vertex set is $P$, and where a pair $p, q$ of points is connected by an edge if $p$ and $q$ appear consecutively along some line $\ell \in L$ (that is, both $p$ and $q$ are incident to $\ell$ and no other point from $P$ lies on the line segment $\overline{pq}$). Using the straight-line drawing induced by the arrangement of $L$ we may regard $G$ as a geometric graph with at most $\binom{m}{2}$ crossings.

Every line from $L$ contains $k \geqslant 1$ point(s) from $P$ and contributes $k - 1$ edges to $G$. Hence $|E| = I - m$. If $|E| \leqslant 4n$, then $I \leqslant 4n + m$ and the theorem holds. Otherwise, we can apply the Crossing Lemma to obtain

$$\binom{m}{2} \geqslant \mathrm{cr}(G) \geqslant \frac{|E|^3}{64n^2}$$

and so $I \leqslant 2^{5/3}\, n^{2/3}m^{2/3} + m$. □

**Theorem 4.5.** *The maximum number of unit distances determined by $n$ points in $\mathbb{R}^2$ is at most $5n^{4/3}$.*

*Proof.* Let $P$ denote the given set of $n$ points, and consider the set $C$ of $n$ unit circles centered at the points in $P$. Then the number $I$ of incidences between $P$ and $C$ is exactly twice the number of unit distances between points from $P$.

Define a graph $G = (P, E)$ on $P$ where two vertices $p$ and $q$ are connected by an edge if they appear consecutively along some circle $c \in C$ (that is, $p, q \in c$ and they appear consecutively in the circular sequence of points from $P$ along $c$). Note that $G$ is not necessarily a simple graph as it may contain loops and double edges if a circle from C contains only one point or two points, respectively, from P. By construction G can be regarded as a disjoint union of cycles whose vertices correspond to the incidences between P and C. Therefore, we have $|E| = I$.

Obtain a new graph $G' = (P, E')$ from G by removing all edges along circles from C that contain at most two points from P. Note that $|C| = n$ and that every circle whose edges are removed contributes at most two edges to G. Therefore $|E'| \geqslant |E| - 2n$. In $G'$ there are no loops and no two vertices are connected by two edges along the same circle. Therefore, any two vertices are connected by at most two edges because there are exactly two distinct unit circles passing through any two distinct points in $\mathbb{R}^2$.

Obtain a new graph $G'' = (P, E'')$ from $G'$ by removing one copy of every double edge. Clearly $G''$ is a simple graph with $|E''| \geqslant |E'|/2 \geqslant (|E|/2) - n$. As every pair of circles intersects in at most two points, we have $\mathrm{cr}(G'') \leqslant 2\binom{n}{2} < n^2$.

If $|E''| \leqslant 4n$, then $(|E|/2) - n \leqslant 4n$ and so $I = |E| \leqslant 10n < 10n^{4/3}$ and the theorem holds. Otherwise, by the Crossing Lemma we have

$$n^2 > \mathrm{cr}(G'') \geqslant \frac{|E''|^3}{64n^2}$$

and so $|E''| < 4n^{4/3}$. It follows that $I = |E| < 8n^{4/3} + 2n < 10n^{4/3}$. □

**Theorem 4.6.** *For $A \subset \mathbb{R}$ with $|A| = n \geqslant 3$ we have* $\max\{|A + A|, |A \cdot A|\} \geqslant \frac{1}{4}n^{5/4}$.

*Proof.* Let $A = \{a_1, \dots, a_n\}$. Set $X = A + A$ and $Y = A \cdot A$. We will show that $|X||Y| \geqslant \frac{1}{16}n^{5/2}$, which proves the theorem. Let $P = X \times Y \subset \mathbb{R}^2$ be the set of points whose x-coordinate is in X and whose y-coordinate is in Y. Clearly $|P| = |X||Y|$. Next define a set L of lines by $\ell_{ij} = \{(x, y) \in \mathbb{R}^2 : y = a_i(x - a_j)\}$, for $i, j \in \{1, \dots, n\}$. Clearly $|L| = n^2$.

On the one hand, every line $\ell_{ij}$ contains at least n points from P because for $x_k = a_j + a_k \in X$ and $y_k = a_i(x_k - a_j) = a_i a_k \in Y$ we have $(x_k, y_k) \in P \cap \ell_{ij}$, for $k \in \{1, \dots, n\}$. Therefore the number I of incidences betwen P and L is at least $n^3$.

On the other hand, by the Szemerédi-Trotter Theorem we have $I \leqslant 2^{5/3}|P|^{2/3}n^{4/3} + 4|P| + n^2$. Combining both bounds we obtain

$$n^3 \leqslant 2^{5/3}|P|^{2/3}n^{4/3} + 4|P| + n^2.$$

Hence either $4|P| + n^2 \geqslant \frac{n^3}{2}$, which implies $|P| \geqslant \frac{1}{16}n^{5/2}$, for $n \geqslant 3$; or $2^{5/3}\,|P|^{2/3}n^{4/3} \geqslant \frac{n^3}{2}$ and thus

$$|P|^{2/3} \geqslant \frac{n^3}{2 \cdot 2^{5/3}\, n^{4/3}} = \left(\frac{n^5}{256}\right)^{1/3} \Longrightarrow |P| \geqslant \frac{n^{5/2}}{16}.$$

□

**Exercise 4.7.** *Consider two edges e and f in a topological plane drawing so that e and f cross at least twice. Prove or disprove: There exist always two distinct crossings x and y of e and f so that the portion of e between x and y is not crossed by f and the portion of f between x and y is not crossed by e.*

**Exercise 4.8.** *Let* $G$ *be a graph with* $n \geqslant 3$ *vertices,* $e$ *edges, and* $\mathrm{cr}(G) = e - (3n - 6)$*. Show that in every drawing of* $G$ *with* $\mathrm{cr}(G)$ *crossings, every edge is crossed at most once.*

**Exercise 4.9.** *Consider the abstract graph* $G$ *that is obtained as follows: Start from a plane embedding of the* 3*-dimensional (hyper-)cube, and add in every face a pair of (crossing) diagonals. Show that* $\mathrm{cr}(G) = 6 < \overline{\mathrm{cr}}(G)$*.*

**Exercise 4.10.** *A graph is* 1-planar *if it can be drawn in the plane so that every edge is crossed at most once. Show that a* 1*-planar graph on* $n \geqslant 3$ *vertices has at most* $4n - 8$ *edges.*

**Exercise 4.11.** *Show that the bound from the Crossing Lemma is asymptotically tight: There exists a constant* $c$ *so that for every* $n, e \in \mathbb{N}$ *with* $e \leqslant \binom{n}{2}$ *there is a graph with* $n$ *vertices and* $e$ *edges that admits a plane drawing with at most* $ce^3/n^2$ *crossings.*

**Exercise 4.12.** *Show that the maximum number of unit distances determined by* $n$ *points in* $\mathbb{R}^2$ *is* $\Omega(n \log n)$*. Hint: Consider the hypercube.*

## Questions

14. *What is the crossing number of a graph? What is the rectilinear crossing number?* Give the definitions and examples. Explain the difference.
15. *For a nonplanar graph, the more edges it has, the more crossings we would expect. Can you quantify such a correspondence more precisely?* State and prove Lemma 4.2 and Theorem 4.3 (The Crossing Lemma).
16. *Why is it called "Crossing Lemma" rather than "Crossing Theorem"?* Explain at least two applications of the Crossing Lemma, for instance, your pick out of the theorems 4.4, 4.5, and 4.6.

## References


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