Chapter 11

Counting

We consider the problems of counting (i) simplices spanned by $d+1$ out of n points in $\mathbb{R}^d$ that contain a query point; and (ii) facets of the convex hull of n points in $\mathbb{R}^d$. These two problems are closely related by yet another duality called *Gale Duality*.

Counting refers to *extremal counting* (given only the parameters, what is the maximum/minimum possible number of the considered object), and to *algorithmic counting* (given a concrete input, compute the number of the considered object). Sometimes we are also interested in *enumeration* (given a concrete input, produce all objects under consideration).

Here are a few notational conventions: $\mathbf{0} := (0, 0, \ldots, 0)$ is the origin in the considered ambient space $\mathbb{R}^d$. $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $[n] := \{1, 2, \ldots, n\}$. $\binom{S}{k}$ denotes the set of all k-element subsets of a given set S. Finally, **Checkpoints** are usually simple quizzes to check your understanding of definitions or notions, to be answered perhaps in a minute or two if you truly absorbed the material.

It will be useful to remember

$$\sum_{i=0}^{n-1} \binom{i}{k-1} = \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} .$$

It does not hurt to recapitulate the combinatorial interpretation of these identities. Recall that $\binom{n}{k} = \left|\binom{[n]}{k}\right|$ is the number of k-element subsets of $[n]$. Every subset $A \in \binom{[n]}{k}$ has a unique maximum element, and we *charge* A to that element.[1] Conversely, if $j \in [n]$ is charged by $A = \binom{[n]}{k}$, then the set A has to consist of j together with a $(k-1)$-element subset of $[j-1]$. In other words, j is charged exactly $\binom{j-1}{k-1}$ times. Therefore,

$$\binom{n}{k} = \sum_{j=1}^{n} \binom{j-1}{k-1} ,$$

[1]At the end of the day, "charging" here means nothing but the mapping $A \mapsto \max(A)$. We also say that $\max(A)$ is "charged by" or "witnesses" A. These are established counting jargon. Often, an object is charged multiple times.

which proves the first identity. For the second identity, we discriminate sets $A \in \binom{[n]}{k}$ by whether n is selected or not. (Rephrasing in our counting jargon, we charge A to "true" if $n \in A$, and to "false" otherwise.) There are $\binom{n-1}{k-1}$ sets where n is selected, and $\binom{n-1}{k}$ sets where it is not. This shows

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} .$$

11.1 Introduction

Consider a set $P \subseteq \mathbb{R}^d$ and a point $q \in \mathbb{R}^d$. We call a set $A \in \binom{P}{d+1}$ a q-**embracing simplex** if $q \in \mathrm{conv}(A)$. The **simplicial depth** of point q is the number of q-embracing simplices; that is

$$\mathrm{sd}_q(P) := \left| \left\{ A \in \binom{P}{d+1} \;:\; q \in \mathrm{conv}(A) \right\} \right| .$$

Note that when specialized to $\mathbb{R}^1$, a median of P is exactly a point of maximum simplicial depth. So this notion, among others, is a possible response to the search for a higher-dimensional counterpart of medians. We will investigate questions like:

- *How large can the simplicial depth of q be, in any set of n points in general position?*
- *How efficiently can we compute the simplicial depth of a point?*

For these questions, we may assume without loss of generality that $q = 0$, as we could translate all the points rigidly with q while preserving the embracing property.

A second direction we want to explore is the complexity of polytopes in general dimension d:

- *How many facets can a polytope defined by n points have? How few?*
- *Given n points, how efficiently can we compute the number of facets of its convex hull? Can we do that asymptotically faster than enumerating these facets (which is a hard enough problem per se)?*

A small reminder as has been reiterated in Chapter 9: The number of facets are linear in n for $d = 2, 3$, and can be quadratic in n for $d = 4$. For general dimension the superlinear growth continues, and we will see what the right bounds are.

These two directions about simplicial depth and number of facets of a polytope are very closely related; in a sense that we will make very explicit later (via the so-called Gale Duality), it is the same question. But for the moment, let us focus on the simplicial depth view.

11.2 Embracing Sets in the Plane

In this section we investigate simplicial depth in the plane $\mathbb{R}^2$. As we mentioned, we may assume $q = 0$. First we generalize embracing simplices (which are triangles in $\mathbb{R}^2$) to *embracing sets*, relaxing the constraint on cardinality. This is not only a natural step to take, but also integral to the argument even if we were interested in simplicial depth only.

Consider a set P of n points in $\mathbb{R}^2$, with $0 \notin P$ and $P \dot\cup \{0\}$ in general position (no three collinear points). This setting will be implicitly assumed throughout the section. For $k \in \mathbb{N}_0$, we define

$$e_k = e_k(P) := \left| \left\{ A \in \binom{P}{k} \,:\, 0 \in \operatorname{conv}(A) \right\} \right|.$$

We call a set $A \in \binom{P}{k}$ with $0 \in \operatorname{conv}(A)$ an **embracing** k-**set**. When $|A| = 3$, it is called an **embracing triangle**.

Checkpoint 11.1. $e_0 = e_1 = e_2 = 0$ *by general position.* $e_3 = \mathrm{sd}_0(P)$ *is the simplicial depth of* 0 *in* P*. And* $e_n \in \{0, 1\}$ *indicates whether* $0 \in \operatorname{conv}(P)$.

We start a general investigation of the vector $\vec{e} = (e_0, e_1, \ldots, e_n) \in \mathbb{N}_0^{n+1}$. Bounds and algorithms will follow easily, but you need to be patient, until it becomes apparent how everything fits together nicely—reward will come. As a preparation consider real vectors $\vec{x}_{0..n-3} = (x_0, x_1, \ldots, x_{n-3})$, $\vec{y}_{0..n-2}$ and $\vec{z}_{0..n-1}$ satisfying

$$e_k = \sum_{i=0}^{n-3} \binom{i}{k-3} x_i \quad \text{for all } k \geqslant 3, \tag{11.2}$$

$$e_k = \sum_{i=0}^{n-2} \binom{i}{k-2} y_i \quad \text{for all } k \geqslant 2, \tag{11.3}$$

$$e_k = \sum_{i=0}^{n-1} \binom{i}{k-1} z_i \quad \text{for all } k \geqslant 1. \tag{11.4}$$

Observe that $\vec{x}_{0..n-3}$ exists and is uniquely determined by $\vec{e}_{3..n}$, since

$$\begin{array}{rcl} e_n = \binom{n-3}{n-3} x_{n-3} & \Longrightarrow & x_{n-3} = e_n \\ e_{n-1} = \binom{n-4}{n-4} x_{n-4} + \binom{n-3}{n-4} x_{n-3} & \Longrightarrow & x_{n-4} = e_{n-1} - (n-3)\underbrace{x_{n-3}}_{e_n} \\ \vdots & & \vdots \end{array}$$

Similarly, this works for $\vec{y}_{0..n-2}$ and $\vec{z}_{0..n-1}$. Thus we have

$$\vec{e}_{3..n} \xleftrightarrow[\text{each other}]{\text{determine}} \vec{x}_{0..n-3}, \qquad \vec{e}_{2..n} \xleftrightarrow[\text{each other}]{\text{determine}} \vec{y}_{0..n-2}, \qquad \vec{e}_{1..n} \xleftrightarrow[\text{each other}]{\text{determine}} \vec{z}_{0..n-1}\ .$$

Note that these facts hold for any vector $\vec{e}$, as we have not used any property of the specific $\vec{e}$ we are interested in. They simply describe some possible transformations for any given vector, although it is by no means clear how they should help...

11.2.1 Adding a Dimension

Another step that comes across unmotivated: Lift the point set P vertically to a set P' in space, *arbitrarily*, with the only requirement being general position (no four points from P' on a common plane). For example, we *may* choose the parabolic lifting map $(x, y) \mapsto (x, y, x^2 + y^2)$; but stay flexible! Let us denote the lifting by

$$P \ni \quad q = (x, y) \mapsto q' = (x, y, z(q)) \quad \in P'.$$

For an embracing triangle $\Delta = \{p, q, r\}$ in the plane, let β_Δ be the number of lifted points in P' strictly below the plane containing $\Delta' = \{p', q', r'\}$. (Just to avoid confusion: β_Δ clearly depends on the choice of the lifting.) Let

$$h_i := \text{ the number of embracing triangles } \Delta \text{ with } \beta_\Delta = i.$$

Checkpoint 11.5. $\sum_{i=0}^{n-3} h_i = e_3$.

Let us recall here that we are assuming general position for $P \dot\cup \{0\}$.

Lemma 11.6. $0 \in \operatorname{conv}(P) \iff h_0 = h_{n-3} = 1$.

Proof. ($\Leftarrow$) That's obvious, since $h_0 = 1$ says that there exists an embracing triangle, and in particular 0 is in the convex hull of P.

($\Rightarrow$) If $0 \in \operatorname{conv}(P)$ then the z-axis (i.e. the vertical line through 0 in $\mathbb{R}^3$) intersects $\operatorname{conv}(P')$. Due to general position, it intersects exactly two facets, both of which are triangles. The bottom one Δ_0' has no point in P' below its supporting plane, thus $\beta_{\Delta_0} = 0$. The top one Δ_1' has no point in P' above its supporting plane, hence all but the three points defining Δ_1' are below, that is $\beta_{\Delta_1} = n - 3$. Clearly both Δ_0 and Δ_1 are embracing, so $h_0, h_{n-3} \geqslant 1$.

On the other hand, any embracing triangle $\Delta \in \binom{P}{3}$ counted by h_0 (respectively h_{n-3}) has all other points in P' above (respectively below) Δ', hence Δ' must give rise to a facet. In addition it must be hit by the z-axis by the embracing property. But Δ_0' and Δ_1' are the only candidates, so $h_0 = h_{n-3} = 1$. □

Consider an embracing k-set $A \in \binom{P}{k}$ and its lifting A'. As observed before, the z-axis intersects the boundary of $\operatorname{conv}(A')$ in two facets. Consider the top facet—it is given by the lifting of some embracing triangle $\Delta \in \binom{P}{3}$. We say that this Δ *witnesses* (the embracing property of) A. How many embracing k-sets does Δ witness?

For Δ to witness an embracing k-set B, we must have $\Delta \subseteq B$ and the remaining $k-3$ points in $B \setminus \Delta$ are chosen so that $B' \setminus \Delta'$ is below the plane spanned by Δ'. Hence Δ witnesses exactly $\binom{\beta_\Delta}{k-3}$ embracing k-sets. It follows that

$$e_k = \sum_{\Delta \in \binom{P}{3} \text{ embracing}} \binom{\beta_\Delta}{k-3} = \sum_{i=0}^{n-3} \binom{i}{k-3} h_i. \tag{11.7}$$

Note that this is exactly the relation (11.2) (with h_i replacing x_i). We thus have

$$\vec{e}_{3..n} \xleftrightarrow[\text{each other}]{\text{determine}} \vec{h}_{0..n-3} := (h_0, h_1, \ldots, h_{n-3})$$

and therefore the vector $\vec{h}_{0..n-3}$ is independent of the lifting we chose, i.e. $h_i = h_i(P)$.

A few properties emerge. First, note that $\vec{h}$ (consisting of nonnegative integers no larger than $\binom{n}{3} = O(n^3)$, or $O(\log n)$ bits) is a compact way of representing $\vec{e}$ (consisting of integers potentially exponential in n, or $\Omega(n)$ bits). Also, since it is easy to compute the vector $\vec{h}$ in $O(n^4)$ time[2], we can compute each entry of e_k in $O(n^4)$ time.

Second, the independence of the vector $\vec{h}$ from the chosen lifting allows quite simple proofs of properties of $\vec{h}$: You can choose the lifting! If you can make a property of $\vec{h}$ hold for a chosen lifting, then it will be true for all liftings. Keep this in mind when solving the following exercise.

Exercise 11.8. *Show that*

$$h_0 = 1 \iff 0 \in \operatorname{conv}(P) \iff h_i \geqslant 1 \textit{ for } 0 \leqslant i \leqslant n-3.$$

Now, don't hesitate to use the assertion of this exercise and relation (11.7) for the following exercise.

Exercise 11.9. *Assume* $0 \in \operatorname{conv}(P)$. *What is the minimal possible value of* e_3 *in terms of* $n := |P|$*? (Note that this is a quantified version of Carathéodory's Theorem for* $\mathbb{R}^3$*.) Generally, what is the minimal possible value of* e_k*,* $3 \leqslant k \leqslant n$*?*

Exercise 11.10. *Let* $P \subset \mathbb{R}^2$ *be a set of* n *points in general position (with the origin). What does* $\sum_{i=0}^{n-3} 2^i h_i$ *count?*

Exercise 11.11. *Let* $P \subset \mathbb{R}^2$ *be a set of* n *points in general position (with the origin) and assume* $0 \in \operatorname{conv}(P)$. *Recall that* e_k *denotes the number of embracing* k*-sets. Show that* $\sum_{k=3}^{n}(-1)^k e_k = -1$*. (Hint: Plug in the relation* $e_k = \sum_{i=0}^{n-3}\binom{i}{k-3}h_i$ *in this sum and simplify.)*

In a next step we show that the vector $\vec{h}$ is symmetric.

Lemma 11.12. $h_i = h_{n-3-i}$.

Proof. Define $\widehat{h}_i$ in the same way as h_i, except that you count the points *above* (instead of below) the plane through the lifting of an embracing triangle. Note that $\widehat{h}_i = h_{n-3-i}$ by definition. On the other hand, with the same witness argument as before we derive

$$e_k = \sum_{i=0}^{n-3}\binom{i}{k-3}\widehat{h}_i,$$

and therefore $h_i = \widehat{h}_i = h_{n-3-i}$. □

[2] With some tricks from computational geometry, in $O(n^3)$ time.

Hence the vector $\vec{h}_{0..n-3}$ is determined by its first half $h_0, h_1, \ldots, h_{\lfloor (n-3)/2 \rfloor}$.

Exercise 11.13. *Let $P \subset \mathbb{R}^2$ be a set of n points in general position (with the origin). Show that e_3 (the number of embracing triangles) and e_4 (the number of embracing 4-sets) are related by $(n-3)e_3 = 2e_4$.*

Exercise 11.14. *Show that if $|P| = 6$, then e_3 determines $e_{3..6}$. How?*

Exercise 11.15. *Show that if $|P|$ is even then e_3 is even.*

11.2.2 The Upper Bound

We have seen in one of the exercises how the relation between $\vec{e}$ and $\vec{h}$ can be useful in proving lower bounds on the e_k's. We need two lemmas towards a proof of upper bounds. The first lemma states that removing a point from P cannot increase h_j.

Lemma 11.16. *For all $j \in \mathbb{N}_0$ and all $q \in P$, we have $h_j(P \setminus \{q\}) \leqslant h_j(P)$.*

Proof. What changes happen to h_j as we remove a point q from P?

- We *lose* those embracing triangles Δ with $\beta_\Delta = j$ (before removal) such that q' is in or below Δ'.
- We *keep* those embracing triangles Δ with $\beta_\Delta = j$ such that q' is above Δ'.
- We *gain* those embracing triangles Δ with $\beta_\Delta = j+1$ such that q' is below Δ'.

Now lift q' vertically above all planes defined by three points in $P' \setminus \{q'\}$. It does not change the values h_i as $\vec{h}$ is independent of the lifting, but eliminates the "gain" case. This gives the lemma. □

Lemma 11.17. *For all $j \in \mathbb{N}_0$ we have*

$$\sum_{q \in P} h_j(P \setminus \{q\}) \;=\; (n-j-3)\, h_j(P) \;+\; (j+1)\, h_{j+1}(P).$$

Proof. Fix an arbitrary lifting. A contribution to $\sum_{q\in P} h_j(P \setminus \{q\})$ can come only from triangles Δ with $\beta_\Delta = j$ or $\beta_\Delta = j+1$ (relative to the complete point set P).

- If $\beta_\Delta = j$, then Δ' remains a triangle with j points below after removing q iff q is one of the $(n-3-j)$ points above.
- If $\beta_\Delta = j+1$, then Δ' turns into a triangle with j points below after removing q iff q is one of the $(j+1)$ points below.

Hence the lemma. □

Now we apply the previous Lemma 11.16 to bound

$$\sum_{q \in P} h_j(P \setminus \{q\}) \leqslant n \cdot h_j(P) \;\; =: n \cdot h_j,$$

and with Lemma 11.17 we can derive

$$\begin{aligned}(n-j-3)h_j + (j+1)h_{j+1} &\leqslant n \cdot h_j \\ (j+1)h_{j+1} &\leqslant (j+3)h_j \\ h_{j+1} &\leqslant \frac{j+3}{j+1} h_j \;.\end{aligned}$$

This bound can be iterated until we reach h_0:

$$h_{j+1} \leqslant \frac{j+3}{j+1} h_j \leqslant \frac{j+3}{j+1}\frac{j+2}{j} h_{j-1} \leqslant \cdots \leqslant \underbrace{\frac{j+3}{j+1}\frac{j+2}{j}\cdots\frac{3}{1}}_{=\binom{j+3}{2}} \underbrace{h_0}_{\leqslant 1} \leqslant \binom{j+3}{2} \;.$$

Theorem 11.18. *Let* P *be a set of* n *points in general position. Then for* $0 \leqslant j \leqslant n-3$ *we have* $h_j = h_{n-3-j}$ *and* $h_j \leqslant \binom{j+2}{2}$. *Consequently* $h_j \leqslant \min\left\{\binom{j+2}{2}, \binom{n-1-j}{2}\right\}$. *Moreover,*

$$e_3 \leqslant \begin{cases} 2\binom{n/2+1}{3} = \frac{n(n^2-4)}{24} & \textit{for } n \textit{ even,} \\ 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2} = \frac{n(n^2-1)}{24} & \textit{for } n \textit{ odd.} \end{cases}$$

Proof. The first part is just a summary of what we have derived so far. For the "moreover" part, we simply plug them into relation (11.7). Suppose first that n is even. Then

$$(h_0, h_1, \ldots, h_{n/2-2}) = (h_{n-3}, h_{n-4}, \ldots, h_{n/2-1})$$

and, therefore,

$$e_3 = \sum_{i=0}^{n-3} h_i = 2\sum_{i=0}^{n/2-2} h_i \leqslant 2\sum_{i=0}^{n/2-2}\binom{i+2}{2} = 2\binom{n/2+1}{3} \;.$$

Second, if n is odd then

$$(h_0, h_1, \ldots, h_{(n-3)/2}) = (h_{n-3}, h_{n-2}, \ldots, h_{(n-3)/2})$$

with $h_{(n-3)/2}$ appearing on both sides. So

$$\begin{aligned} e_3 = \sum_{i=0}^{n-3} h_i &= 2\sum_{i=0}^{(n-3)/2-1} h_i + h_{(n-3)/2} \\ &\leqslant 2\sum_{i=0}^{(n-3)/2-1}\binom{i+2}{2} + \binom{(n+1)/2}{2} \\ &= 2\binom{(n+1)/2}{3} + \binom{(n+1)/2}{2} \;. \end{aligned}$$

□

There are sets where all these bounds are tight, simultaneously. We find it more convenient to substantiate this claim after further considerations.

Exercise 11.19. *Show* $e_3 \leqslant \frac{1}{4}\binom{n}{3} + O(n^2)$. *(That is, asymptotically, at most* $1/4$ *of all triangles embrace the origin.)*

Exercise 11.20. *Try to understand the independence of* $\vec{h}$ *of the actual lifting by observing what happens as you move a single point vertically.*

We have obtained lower and upper bounds in the plane. Before proceeding to better methods for computing the e_k's, we generalize to arbitrary dimension d.

11.3 Embracing Sets in Higher Dimension

It has been announced that our methods easily carry over to higher dimensions. So let us do a quick tour of deriving the bounds analogous to Theorem 11.18. The reader should make sure that indeed all arguments can be generalized. It is a good exercise to recapitulate the proofs.

Let us now assume that $P \subset \mathbb{R}^d$ is a set of n points in general position with the origin, that is, $0 \notin P$ and no $d+1$ points in $P \cup \{0\}$ lie on a common hyperplane. There is no change in the notion of an embracing k-set and of the vector $\vec{e}$, but let us still repeat:

For $k \in \mathbb{N}_0$, we define $e_k = e_k(P) := \left|\left\{A \in \binom{P}{k} : 0 \in \text{conv}(A)\right\}\right|$. We call a set $A \in \binom{P}{k}$ with $0 \in \text{conv}(A)$ an **embracing** k**-set**. When $|A| = d+1$, it is called an **embracing simplex**. We will still use symbol Δ for embracing simplices. Observe that $e_0 = e_1 = \cdots = e_d = 0$ by general position, and that $e_{d+1} = \text{sd}_0(P)$.

We consider a generic vertical lifting from P to $\mathbb{R}^{d+1}$, denoted by $p \mapsto p'$. "Vertical" means we lift along the new dimension; "generic" means that no $d+2$ points in P' lie in a common hyperplane.

If $\Delta \in \binom{P}{d+1}$ is an embracing simplex, then its lifting Δ' affinely spans a hyperplane. We use β_Δ for the number of P' strictly below this hyperplane. We emphasize that β_Δ depends on the lifting chosen.

As before, we define the vector $\vec{h}$ with

$$h_i := \text{ the number of embracing simplices } \Delta \text{ with } \beta_\Delta = i.$$

with the only difference that we now consider embracing simplices rather than triangles.

Checkpoint 11.21. $\sum_{i=0}^{n-(d+1)} h_i = e_{d+1}$.

In the plane, our next lemma was $0 \in P \iff h_0 = h_{n-3} = 1$, where h_{n-3} counted all embracing triangles Δ with all other points below (in the lifting). This number is now $n-(d+1)$, so we get

Lemma 11.22. $0 \in P \iff h_0 = h_{n-(d+1)} = 1$

We can take over the proof we have seen for Lemma 11.6. Essential ingredients were that the x_{d+1}-axis intersects the convex polytope $\text{conv}(P')$ in a non-empty interval. Each of its endpoints is on a facet of the polytope. The supporting hyperplane of one facet has no point in P' below, and the supporting hyperplane of the other facet has no point above (here we use the fact that the intersecting line, the x_{d+1}-axis, is vertical). Hence, these facets are liftings of embracing simplices Δ_0 and Δ_1, respectively, with $\beta_{\Delta_0} = 0$ and $\beta_{\Delta_1} = n - (d+1)$. Via the notion of a witness embracing simplex $\Delta \subseteq A$ of an embracing k-set A, the counterpart of (11.7) reads now

$$e_k = \sum_{\Delta \in \binom{P}{d+1} \text{ embracing}} \binom{\beta_\Delta}{k-(d+1)} = \sum_{i=0}^{n-(d+1)} \binom{i}{k-(d+1)} h_i \,, \tag{11.23}$$

and thus

$$\vec{e}_{d+1..n} \xleftrightarrow[\text{each other}]{\text{determine}} \vec{h}_{0..n-(d+1)}.$$

Hence, h_i is independent of the lifting chosen and we can write $h_i = h_i(P)$. Symmetry of $\vec{h}$ follows readily, as before, by looking at points above instead of below a lifted embracing simplex.

Lemma 11.24. $h_i = h_{n-(d+1)-i}$.

The two lemmas towards the upper bound carry over, with the first identical to what we have seen before, and the second with the constants adapted to the dimension.

Lemma 11.25. *For all* $j \in \mathbb{N}_0$ *and all* $q \in P$*, we have* $h_j(P \setminus \{q\}) \leqslant h_j(P)$*.*

Lemma 11.26. *For all* $j \in \mathbb{N}_0$ *we have*

$$\sum_{q \in P} h_j(P \setminus \{q\}) \;=\; \big(n - j - (d+1)\big)\, h_j(P) \;+\; (j+1)\, h_{j+1}(P).$$

Proof. Fix an arbitrary generic lifting. A contribution to $\sum_{q \in P} h_j(P \setminus \{q\})$ can come only from simplices Δ with $\beta_\Delta = j$ or $\beta_\Delta = j+1$ (relative to the complete set P).

- If $\beta_\Delta = j$, then Δ' remains a simplex with j points below after removing q, iff q is one of the $(n-(d+1)-j)$ points above.
- If $\beta_\Delta = j+1$, then Δ' turns into a simplex with j points below after removing q, iff q is one of the $(j+1)$ points below. □

Again, for the upper bound on the h_i's, just like in the plane, we start with

$$\sum_{q \in P} h_j(P \setminus \{q\}) \leqslant n \cdot h_j(P)$$

by Lemma 11.25, and then continue

$$\begin{aligned}(n-j-(d+1))h_j+(j+1)h_{j+1} &\leqslant n\cdot h_j\\ (j+1)h_{j+1} &\leqslant (j+d+1)h_j\\ h_{j+1} &\leqslant \frac{j+d+1}{j+1}h_j\end{aligned}$$

by Lemma 11.26. Then we iterate this until we reach h_0:

$$h_{j+1}\leqslant\frac{j+d+1}{j+1}h_j\leqslant\cdots\leqslant\underbrace{\frac{j+d+1}{j+1}\,\frac{j+d}{j}\,\cdots\,\frac{d+1}{1}}_{=\binom{j+d+1}{d}}\underbrace{h_0}_{\leqslant 1}\leqslant\binom{j+d+1}{d}.$$

Theorem 11.27. *Let $P\subset\mathbb{R}^d$ be a set of n points in general position. Then for $0\leqslant j\leqslant n-(d+1)$ we have $h_j=h_{n-(d+1)-j}$ and $h_j\leqslant\binom{j+d}{d}$. Consequently $h_j\leqslant\min\left\{\binom{j+d}{d},\binom{n-1-j}{d}\right\}$. Moreover,*

$$e_{d+1}\leqslant\begin{cases}2\binom{(n+d)/2}{d+1} & \text{for } n-d \text{ even,}\\ 2\binom{(n+d-1)/2}{d+1}+\binom{(n+d-1)/2}{d} & \text{for } n-d \text{ odd.}\end{cases}$$

Proof. The first part is just a summary of what we have derived so far. For the "moreover" part, we simply plug them into relation (11.23). For $n-d$ even we have

$$(h_0,h_1,\ldots,h_{(n-d)/2-1})=(h_{n-d-1},h_{n-d-2},\ldots,h_{(n-d)/2})$$

and, therefore,

$$e_{d+1}=\sum_{i=0}^{n-(d+1)}h_i=2\sum_{i=0}^{(n-d)/2-1}h_i\leqslant 2\sum_{i=0}^{(n-d)/2-1}\binom{i+d}{d}=2\binom{(n+d)/2}{d+1}.$$

If $n-d$ is odd then

$$(h_0,h_1,\ldots,h_{(n-(d+1))/2})=(h_{n-3},h_{n-2},\ldots,h_{(n-(d+1))/2})$$

with $h_{(n-(d+1))/2}$ appearing on both sides. So

$$\begin{aligned}e_{d+1}=\sum_{i=0}^{n-(d+1)}h_i &= 2\sum_{i=0}^{(n-(d+1))/2-1}h_i\ +h_{(n-(d+1))/2}\\ &\leqslant 2\sum_{i=0}^{(n-(d+1))/2-1}\binom{i+d}{d}\ +\binom{(n+d-1)/2}{2}\\ &=2\binom{(n+d-1)/2}{d+1}+\binom{(n+d-1)/2}{d}.\end{aligned}$$

□

11.4 Embracing Sets vs. Faces of Polytopes

This section exhibits a duality between points sets of size n. It is very different from polarity and projective duality that you have learnt in previous chapters. Roughly speaking, if $P \subset \mathbb{R}^d$ is dual to $Q \subset \mathbb{R}^{n-d-1}$, then the faces of $\text{conv}(P)$ one-one correspond to the embracing sets of Q.

In order to describe this duality, and then to handle it, we need some handy linear algebra terminology as well as the algebraic rephrasing of our target notions "embracing" and "supporting hyperplane" (for polytope faces). We will approach this smoothly, and I apologize to those who have these matters on top of their head anyway.[3]

11.4.1 Warm-up

Point sequences and matrices. For integers $d, n \in \mathbb{N}_0$, consider a matrix $A \in \mathbb{R}^{n \times d}$. The sequence $S_A = (p_1, p_2, \ldots, p_n)$ of row vectors of A can be interpreted as a sequence of points in $\mathbb{R}^d$ (or strictly speaking $\mathbb{R}^{1 \times d}$, if we want to emphasize that they are row vectors). Vice versa, every sequence of n points in $\mathbb{R}^d$ can be thought of as a matrix $A \in \mathbb{R}^{n \times d}$. Let us say right away that we abandon the general position assumption, at least for the time being. In particular, we allow repetitions in a sequence of points.

We write $\vec{1}$ and $\vec{0}$ for the vector of all 1's and all 0's, respectively. Their dimensions and their being a row or a column will be clear from the context. Hence $\vec{0}$ also represents the origin in the ambient space. Given a vector $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ (row or column), we write $u \geqslant \vec{0}$ if $u_i \geqslant 0$ for all $i = 1, \ldots, m$.

Linear and convex combinations. A linear combination $\lambda_1 p_1 + \cdots + \lambda_n p_n$ of the rows of $A \in \mathbb{R}^{n \times d}$ with coefficients $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{1 \times n}$ can be compactly written as matrix multiplication $\lambda \cdot A \in \mathbb{R}^{1 \times d}$. Here are a few simple observations.

(1) $\frac{1}{n}(\vec{1} \cdot A)$ is the centroid[4] of S_A.

(2) $\vec{1} \cdot A = \vec{0}$ iff $\vec{0}$ is the centroid of S_A. Another way of interpreting $\vec{1} \cdot A = \vec{0}$ is that $\vec{1}$ is orthogonal to all column vectors of A.

(3) $\lambda \cdot A$, with $\lambda \geqslant \vec{0}$ and $\vec{1} \cdot \lambda = 1$, is a convex combination of S_A.

(4) If $\lambda \cdot A = \vec{0}$ with $\vec{0} \neq \lambda \geqslant \vec{0}$, then $\vec{0} \in \text{conv}(S_A)$. The reason is that we can scale such λ to convex coefficients $\lambda' := \frac{1}{\vec{1} \cdot \lambda} \lambda$ which also satisfies $\lambda' \cdot A = \vec{0}$.

Just like the left product $\lambda \cdot A$ denotes a linear combination of the rows of A, the right product $A \cdot \mu$, for $\mu \in \mathbb{R}^d$, denotes a linear combination of the columns of A.

[3] Think of it as a warm-up of your linear-algebra-muscles.
[4] Center of gravity, or the average of the points.

Hyperplanes. An oriented hyperplane in $\mathbb{R}^d$ is represented by a column vector $v \in \mathbb{R}^{d+1}$:

$$H_v := \left\{ x \in \mathbb{R}^d \ : \ (x, -1) \cdot v = \sum_{i=1}^{d} v_i x_i - v_{d+1} = 0 \right\} \text{ for } v = \begin{pmatrix} v_1 \\ \vdots \\ v_{d+1} \end{pmatrix} \neq \vec{0}.$$

Here $(x, -1)$ is the row vector x extended by an extra dimension with coordinate -1. Denote by H_v^+ the closed positive halfspace bounded by H_v. Recall that H_v is a supporting hyperplane of some face of $\mathcal{P}$ if $\mathcal{P} \subseteq H_v^+$.

(1) The vector $\sigma := (A, -\vec{1}) \cdot v \in \mathbb{R}^n$ indicates the relations between the points p_i and the hyperplane H_v:

$$\sigma_i = 0 \iff p_i \in H_v,$$
$$\sigma_i \geqslant 0 \iff p_i \in H_v^+.$$

Here $(A, -\vec{1})$ denotes the matrix in $\mathbb{R}^{n \times (d+1)}$ obtained from A by extending it by an extra column $-\vec{1}$.

(2) $(A, -\vec{1}) \cdot v = \vec{0}$ iff H_v contains all points from S_A.

(3) $(A, -\vec{1}) \cdot v \geqslant \vec{0}$ iff H_v is a supporting hyperplane of some face of $\mathrm{conv}(S_A)$.

We recall that matrix $A \in \mathbb{R}^{n \times d}$ has full rank d iff its columns are independent; that is, there is no $\mu \neq \vec{0}$ with $A \cdot \mu = \vec{0}$. Rephrased geometrically, $\mathrm{rank}(A) = d$ iff there is no hyperplane $H_{(\mu,0)}$ through the origin that contains all points from S_A.

11.4.2 Gale Duality

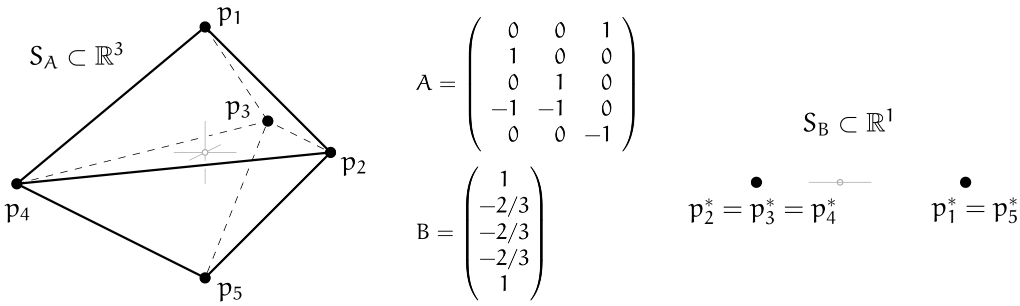
Assume $0 \leqslant d < n$. We are now ready to describe a duality between sequences of n points in $\mathbb{R}^d$ and $\mathbb{R}^{n-d-1}$.

We call a matrix $A \in \mathbb{R}^{n \times d}$ **legal** if $\vec{1} \cdot A = \vec{0}$ and $\mathrm{rank}(A) = d$. What is the geometric interpretation of legality? The first condition says that the origin is the centroid of S_A. In particular, $\mathrm{conv}(S_A)$ contains the origin. Hence $\mathrm{conv}(S_A)$ is a full dimensional polytope by the second condition: otherwise $\mathrm{conv}(S_A)$ is entirely contained in some hyperplane (which has to go through the origin), contradicting $\mathrm{rank}(A) = d$.

Vice versa, if S_A has centroid $\vec{0}$ and $\mathrm{conv}(S_A)$ is full-dimensional, then A is legal. Hence legality is a much weaker assumption than general position!

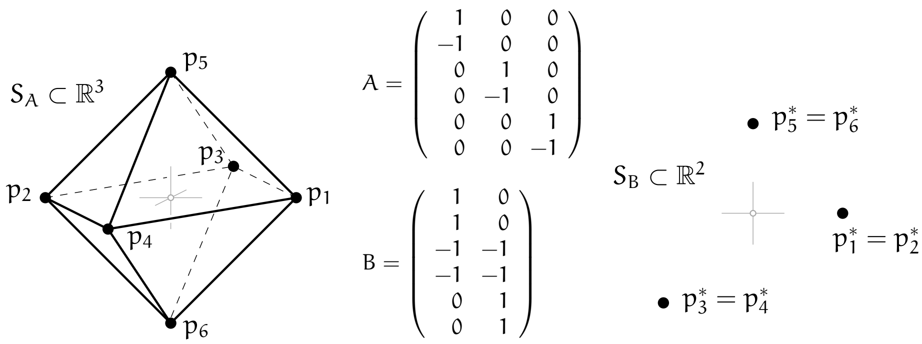
Given legal matrices $A \in \mathbb{R}^{n \times d}$ and $B \in \mathbb{R}^{n \times (n-d-1)}$, we call B an **orthogonal dual** of A, in symbols $A \perp B$, if $A^\top B = 0^{d \times (n-d-1)}$. In other words, all columns of A are orthogonal to all columns of B; as a result, the columns of A span a linear space of dimension d orthogonal to the linear space of dimension $n - d - 1$ spanned by the columns of B, and both spaces are orthogonal to $\vec{1}$ (by the legality condition). Hence for any legal matrix A, we may always find an orthogonal dual B, and it is unique up to linear transformations.

Clearly, $A \perp B \iff B \perp A$.[5] See Figures 11.1 and 11.2 for examples of orthogonal duals and their point sequences.



$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ -2/3 \\ -2/3 \\ -2/3 \\ 1 \end{pmatrix}$$

Figure 11.1: *Point sequences S_A and S_B from orthogonal duals $\mathbb{R}^{5\times 3} \ni A \perp B \in \mathbb{R}^{5\times 1}$.*



$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & -1 \\ -1 & -1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Figure 11.2: *Point sequences S_A and S_B from orthogonal duals $\mathbb{R}^{6\times 3} \ni A \perp B \in \mathbb{R}^{6\times 2}$.*

Lemma 11.28 (Gale Duality). *Let $\mathbb{R}^{n\times d} \ni A \perp B \in \mathbb{R}^{n\times(n-d-1)}$ be legal matrices that are orthogonal duals to each other, and denote $S_A = (p_1, \dots, p_n)$ and $S_B = (p_1^*, \dots, p_n^*)$. For any given $I \subseteq [n]$, consider $F := \{p_i : i \in I\}$ and $\overline{F^*} := \{p_i^* : i \notin I\}$. Then F is contained in a supporting hyperplane of $\operatorname{conv}(S_A)$ if and only if $\vec{0} \in \operatorname{conv}(\overline{F^*})$.*

Proof. Let F lie in a supporting hyperplane. That is, there is a vector $v \in \mathbb{R}^{d+1}$, $v_{1..d} \neq \vec{0}$, such that $\sigma := (A, -\vec{1}) \cdot v \geqslant \vec{0}$ and $\sigma_i = 0$ for all $i \in I$. Note also $\sigma \neq \vec{0}$ since A has full rank. Moreover,

$$\sigma^\top \cdot B = v^\top \cdot \underbrace{(A, -\vec{1})^\top \cdot B}_{0^{(d+1)\times(n-d-1)}} = \vec{0}$$

which implies that $\vec{0}$ can be written as a convex combination of points in S_B. Since only those points in $\overline{F^*}$ contribute positively, we conclude that $\vec{0} \in \operatorname{conv}(\overline{F^*})$.

[5]This convenient symmetry, enforced by the condition $\vec{1} \cdot A^\top = \vec{0}$, is the only difference to the standard Gale transform—apart from expository details.

For the reverse direction, let vector $\lambda \in \mathbb{R}^n$ certify the fact that $\vec{0} \in \operatorname{conv}(\overline{F^*})$. That is, $\lambda^\top \cdot B = \vec{0}$ with $\vec{0} \neq \lambda \geqslant \vec{0}$ and $\lambda_i = 0$ for $i \in I$. Since λ is orthogonal to the columns of B (that's what $\lambda^\top \cdot B = \vec{0}$ says), it is in the linear space spanned by the columns of $(A, -\vec{1})$, and so there is a vector $\nu \in \mathbb{R}^{d+1}$ with $(A, -\vec{1}) \cdot \nu = \lambda$. Hence, ν represents a supporting hyperplane of $\operatorname{conv}(S_A)$ that passes through $\{p_i : \lambda_i = 0\} \supseteq F$. □

Faces in simplicial polytopes. At this point recall (and you probably already did) the discussion about simplicial polytopes. We have seen that they maximize the number of facets for a given number n of vertices. In such a polytope, every k-face where $0 \leqslant k \leqslant d-1$ is a k-simplex and hence has exactly $k+1$ vertices. The convex hull of any subset of these vertices produces a face of the polytope. Therefore, F is the vertex set of an i-face iff $|F| = i+1$ and F is contained in some supporting hyperplane.[6]

Given a d-dimensional polytope $\mathcal{P}$, define the f-**vector** of $\mathcal{P}$ by

$$\vec{f} = \vec{f}(\mathcal{P}) = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{N}_0^{d+1}$$

where f_i is the number of i-faces. (Recall that there is the empty face, which we agree to be -1-dimensional; but we ignore the d-dimensional face, the whole polytope itself). Hence $f_{-1} = 1$, f_0 is the number of vertices of $\mathcal{P}$, and f_{d-1} is the number of facets of $\mathcal{P}$.

Observation 11.29. *If $\mathcal{P}$ is a d-dimensional simplicial polytope with vertex set $V(\mathcal{P})$, then f_i counts the number of $(i+1)$-element subsets of $V(\mathcal{P})$ that are contained in a supporting hyperplane of $\mathcal{P}$.*

We are ready to employ Gale Duality.

Lemma 11.30. *Let $A \in \mathbb{R}^{n\times d}$ be a legal matrix, whose rows S_A encode n points in $\mathbb{R}^d$ in general position, so that $\operatorname{conv}(S_A)$ is a d-dimensional simplicial polytope with f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$. Suppose that the legal matrix $B \in \mathbb{R}^{n\times(n-d-1)}$ is an orthogonal dual of A, whose rows S_B encode n points in $\mathbb{R}^{n-d-1}$ in general position with the origin. Then*

$$f_{i-1} = e_{n-i}(S_B)$$

for all $0 \leqslant i \leqslant d$. In particular, $f_{-1} = e_n = 1$, the number of vertices is $f_0 = e_{n-1}$, and the number of facets is $f_{d-1} = e_{n-d} = \mathrm{sd}_0(S_B)$.

Proof. Denote $S_A = (p_1, \ldots, p_n)$ and $S_B = (p_1^*, \ldots, p_n^*)$. Since $\operatorname{conv}(S_A)$ is simplicial, f_{i-1} counts the number of $I \in \binom{[n]}{i}$ such that the points $\{p_i : i \in I\}$ are contained in a supporting hyperplane (Observation 11.29). By Gale duality Lemma 11.28, this happens iff $\vec{0} \in \operatorname{conv}\{p_i^* : i \notin I\}$, namely $\{p_i^* : i \notin I\}$ is an embracing $(n-i)$-set of S_B. □

[6] For all of this it is important that the polytope is simplicial. Think of a 3-dimensional cube: The facets, which are 2-faces, have vertex sets of size 4, and 3-element subsets of these 4-element sets are *not* vertex sets of any face.

In order to carry the upper bounds for $\vec{e}$ over $\vec{f}$, it remains to ensure that for every simplicial polytope, the conditions of the lemma can be achieved effectively. To this end, we start with a simplicial polytope, translate its vertices rigidly so that $\vec{0}$ becomes the centroid, and then perturb the vertices into general position with $\vec{0}$, without changing the face lattice and, in particular, the f-vector. That this is possible needs a not too difficult careful argument, which we sweep under the rug here. In fact, under these assumptions, an orthogonal dual of the vertices is also a set in general position with the origin (see Exercise 11.34).

Finally, it comes the bound on the number of faces, first shown by McMullen in 1970.

Theorem 11.31 (Upper Bound Theorem). *Let $\mathcal{P}$ be a simplicial d-dimensional polytope with n vertices and f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$. Then there is a vector $\vec{h} = (h_0, h_1, \ldots, h_d) \in \mathbb{N}_0^{d+1}$ such that*

$$f_{i-1} = \sum_{j=0}^{d} \binom{d-j}{i-j} h_j \quad \textit{with} \quad h_j = h_{d-j} \leqslant \binom{j+n-d-1}{j} \textit{ for all } j.$$

In particular,

$$f_{d-1} \leqslant \left\{ \begin{matrix} 2\binom{n-(d+1)/2}{(d-1)/2} & \textit{for } d \textit{ odd} \\ 2\binom{n-d/2-1}{d/2-1} + \binom{n-d/2-1}{d/2} & \textit{for } d \textit{ even} \end{matrix} \right\} = O\left(n^{\lfloor d/2 \rfloor}\right)$$

The proof of the theorem is just a transformation of Theorem 11.27 via Lemma 11.30. Let us first check the bounds for f_{d-1} for the values we are familiar with: For $d = 2$ we get $f_1 \leqslant 2\binom{n-2}{0} + \binom{n-2}{1} = 2 + n - 2 = n$, and for $d = 3$ we get $f_2 \leqslant 2\binom{n-2}{1} = 2n - 4$, which are both the values to be expected. For $d = 4$, we see that the upper bound $f_3 \leqslant 2\binom{n-3}{1} + \binom{n-3}{2} = \frac{n(n-3)}{2}$ grows quadratically, which confirms that the lower bound in Section 9.7 is asymptotically tight.

Proof. With the translation and perturbation mentioned earlier, we may assume that $V(\mathcal{P})$ has centroid $\vec{0}$ and is in general position with $\vec{0}$. Let $A \perp B$ with S_A an ordering of $V(\mathcal{P})$. Denote $d^* := n - d - 1$. Applying the theory of embracing sets on S_B in the dual space $\mathbb{R}^{d^*}$, we know there is a vector $\vec{h} = (h_0, h_1, \ldots, h_{n-(d^*+1)})$ such that

$$e_k(S_B) = \sum_{j=0}^{n-(d^*+1)} \binom{j}{k-(d^*+1)} h_j \text{ and } h_j = h_{n-(d^*+1)-j} = h_{d-j}.$$

and $h_j \leqslant \binom{j+d^*}{d^*} = \binom{j+n-d-1}{n-d-1} = \binom{j+n-d-1}{j}$. Hence via Lemma 11.30,

$$f_{i-1} = e_{n-i} = \sum_{j=0}^{n-(d^*+1)} \binom{j}{n-i-(d^*+1)} h_j$$
$$= \sum_{j=0}^{d} \binom{j}{d-i} h_j = \sum_{j=0}^{d} \binom{j}{d-i} h_{d-j} = \sum_{j=0}^{d} \binom{d-j}{d-i} h_j = \sum_{j=0}^{d} \binom{d-j}{i-j} h_j.$$

In particular, with the bound in Theorem 11.27, we obtain

$$f_{d-1} = e_{n-d} = e_{d^*+1} \leqslant \left\{ \begin{array}{rl} 2\binom{(n+d^*)/2}{d^*+1} & \text{for } n-d^* \text{ even} \\ 2\binom{(n+d^*-1)/2}{d^*+1} + \binom{(n+d^*-1)/2}{d^*} & \text{for } n-d^* \text{ odd} \end{array} \right\}$$
$$= \left\{ \begin{array}{rl} 2\binom{n-(d+1)/2}{(d-1)/2} & \text{for } d \text{ odd} \\ 2\binom{n-d/2-1}{d/2-1} + \binom{n-d/2-1}{d/2} & \text{for } d \text{ even} \end{array} \right\} = O\left(n^{\lfloor d/2 \rfloor}\right) \qquad \square$$

Tightness. The bounds in the Upper Bound Theorem are tight, for the whole f-vector. A family of polytopes that attain this bound are quite easy to describe: the so-called *cyclic polytope* is the convex hull of $\{(t, t^2, \ldots, t^d) \in \mathbb{R}^d : t = 1, 2, \ldots, n\}$. These polytopes have the property that for all $i \leqslant \lfloor \frac{d}{2} \rfloor$, every i-element subset of the vertices form an $(i-1)$-face. For example, when $d = 4$, all pairs of its vertices are connected by an edge. But we skip the proof that such polytopes have the prescribed number of faces in various dimensions.

The beauty of the theorem goes much beyond supplying an upper bound. Many facts known about polytopes follow now quite naturally.

Dehn-Sommerville relations. The symmetry $h_j = h_{d-j}$ for $0 \leqslant j \leqslant d$ is also called *Dehn-Sommerville relations*. Originally they are formulated in terms of the f-vector, but Sommerville later restated them in the current compact form in terms of the h-vector.

To recover the original form, we recall from the Upper Bound Theorem that $f_{i-1} = \sum_{j=0}^{d} \binom{d-j}{i-j} h_j$. It is not hard to derive an inversion formula $h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}$ just like what we did in Exercise 11.11. The original Dehn-Sommerville relations simply replace both sides of $h_j = h_{d-j}$ by the f-expressions.

Let us discuss an important consequences. Specializing with $j := d$ we get

$$\begin{aligned} 1 = h_d &= \sum_{i=0}^{d} (-1)^{d-i} f_{i-1} \\ &= f_{d-1} - f_{d-2} + f_{d-3} - \cdots + (-1)^{d-1} f_0 + (-1)^d, \end{aligned}$$

which is exactly the Euler-Poincaré Formula that we saw in Chapter 9.

More formulas of the type are $2f_{d-2} = df_{d-1}$, which can be easily obtained directly by double-counting.

The usual proof. The "usual proof" of the Upper Bound Theorem does not take the detour to the Gale dual. Instead, the h-vector is defined directly for a simplicial polytope $\mathcal{P} \subset \mathbb{R}^d$. The ingredients of the proof are similar, actually the same as we saw translated to the Gale Dual. Apart from the original paper by McMullen, see for example the book by Ziegler [?] for this version of the proof.

Exercise 11.32. *Let* $A \in \mathbb{R}^{n\times d}$ *and* $B \in \mathbb{R}^{n\times(n-d-1)}$ *be legal matrices with* $A \perp B$*, such that* S_A *and* S_B *are in general position with the origin (in particular, all points are distinct).*

(i) Suppose that all elements in S_A *are extreme, i.e. vertices of* $\operatorname{conv}(S_A)$*. What does this translate to for the embracing sets of* S_B*?*

(ii) Suppose $f_{i-1}(\operatorname{conv}(S_A)) = \binom{n}{i}$*. What does this translate to for the embracing sets of* S_B*?*

Exercise 11.33. *Show that a simplicial* d*-dimensional polytope* $\mathcal{P}$ *with* $d+2$ *vertices has always* $k(d+2-k)$ *facets, for some* $2 \leqslant k \leqslant d$*.*

Exercise 11.34. *Let* $A \in \mathbb{R}^{n\times d}$ *and* $B \in \mathbb{R}^{n\times(n-d-1)}$ *be legal matrices with* $A \perp B$*. Suppose* $S_A = (p_1,\ldots,p_n)$*,* $S_B = (p_1^*,\ldots,p_n^*)$ *and let* $I \subseteq [n]$*. Note that we do not assume general position beyond the legality of* A *and* B*; in particular,* S_A *and* S_B *may contain repeated points.*

(i) Suppose $|I| = d+1$*, and points* $\{p_i\}_{i\in I}$ *lie in a common hyperplane. What does this translate to for* S_B*?*

(ii) Suppose $|I| = d$*, and points* $\{p_i\}_{i\in I}$ *lie in a common hyperplane with the origin* $\vec{0} \in \mathbb{R}^d$*. What does this translate to for* S_B*?*

(iii) Show that S_A *is generic iff* S_B *is generic. Here we call a sequence* $(p_i)_{i=1}^n$ *of points in* $\mathbb{R}^d$ generic *if it does not contain* $\vec{0}$ *and no* $d+1$ *points in* $\{p_i\}_{i=1}^n \cup \{\vec{0}\}$ *lie in a common hyperplane.*

Exercise 11.35. *Let* $A \in \mathbb{R}^{n\times d}$ *and* $B \in \mathbb{R}^{n\times(n-d-1)}$ *be legal matrices with* $A \perp B$*, such that* S_A *and* S_B *are in general position with the origin (in particular, all points are distinct). Suppose* n *is even.*

We call a vector $\lambda \in \mathbb{R}^n$ balanced*, if no entry of* λ *is* 0*, and there is the same number of positive and negative entries in* λ*. We call* (Q^+, Q^-) *a feasible equipartition of* $S_A = (p_1,\ldots,p_n)$ *if there is a balanced vector* λ *such that* $\lambda \cdot A = \vec{0}$ *and* $Q^+ = \{p_i : \lambda_i > 0\}$ *and* $Q^- = \{p_i : \lambda_i < 0\}$*. What do these feasible equipartitions translate to for the points of* S_B*?*

11.5 Faster Counting in the Plane

For $q \in P$, call the directed segment $\overrightarrow{0q}$ an i-*edge* if there are i points from P lying to its left. Let $\ell_i = \ell_i(P)$ be the number of i-edges of P.

Checkpoint 11.36. $\sum_i \ell_i = n$*. What is the vector* $\vec{\ell} = (\ell_0, \ell_1, \ldots, \ell_{n-1})$ *for the case* $0 \notin \operatorname{conv}(P)$*?*

For every nonempty subset $A \subseteq P$ with $0 \notin \operatorname{conv}(A)$, there is a left tangent and a right tangent to $\operatorname{conv}(A)$ from 0. We charge A to that right tangent point $q \in A$. How many sets $A \in \binom{P}{k}$ with $0 \notin \operatorname{conv}(A)$ charges to this particular point q?

Checkpoint 11.37. *The answer is* $\binom{i}{k-1}$ *if* $\overrightarrow{0q}$ *is an* i*-edge.*

Hence, for $1 \leqslant k \leqslant n$ we have

$$e_k = \underbrace{\binom{n}{k}}_{\text{all } k\text{-sets}} - \underbrace{\sum_{i=0}^{n-1} \binom{i}{k-1} \ell_i}_{\text{non-embracing } k\text{-sets}} = \sum_{i=0}^{n-1} \binom{i}{k-1} (1 - \ell_i) \, . \tag{11.38}$$

As a remark, this fits in the relation (11.4) with $z_i = 1 - \ell_i$, so the numbers ℓ_i satisfying (11.38) are unique.

Exercise 11.39. *Show that* $\ell_i = \ell_{n-1-i}$. *(Hint: Wonder why "left" and not "right".)*

Theorem 11.40. *In the plane, the simplicial depth* $\mathrm{sd}_q(P)$ *can be computed in* $O(n \log n)$ *time, provided* $P \,\dot\cup\, \{q\}$ *is in general position.*

Proof. By translating the points appropriately, we may assume $q = 0$. Then we compute the vector $\vec{\ell}$ in $O(n \log n)$ time. For that we rotate a directed line around 0, starting with the horizontal line, say. We always maintain the number of points left of this line, and update this number whenever we sweep over a point $q \in P$. The q may lie ahead of 0 or behind it; depending on this the number increases or decreases by one, respectively. After a rotation by 180 degrees, we have collected the "number of points to the left" for every $q \in P$. The rotation can be implemented in discrete events; all we need is to sort the points by angle around 0, which takes $O(n \log n)$ time. The initialization of the "number to the left" costs $O(n)$ time, and each update costs $O(1)$ time. This gives $O(n \log n)$ altogether. Finally, from the vector $\vec{\ell}$, we recover the simplicial depth $\mathrm{sd}_q(P) = e_3$ via equation (11.38). □

Similarly, all entries e_k, $1 \leqslant k \leqslant n$, can be computed based on the vector $\vec{\ell}$ using (11.38). However, keep in mind that the binomial coefficients involved in the sum can be large (up to $\Theta(n)$-bit).

Given (11.38), showing that the upper bound in Theorem 11.18 is tight is actually easy. Consider the set of vertices P of a regular n-gon (n odd) centered at 0, then $\ell_{(n-1)/2} = n$ and all other ℓ_i's vanish. Therefore,

$$e_3 = \binom{n}{3} - \binom{(n-1)/2}{2} n = \frac{n(n^2-1)}{24} \, ,$$

thus the case of n odd is tight in Theorem 11.18.

For n even, consider the vertices of a regular n-gon centered at 0, and let P be a slightly perturbed set of the vertices so that $P \cup \{0\}$ is in general position. For every $q \in P$, the directed segment $\overrightarrow{0q}$ is either an $(n/2-1)$-edge or an $(n/2)$-edge. Interestingly, because of the symmetry of the $\vec{\ell}$, we immediately know that $\ell_{n/2-1} = \ell_{n/2} = n/2$ and all other ℓ_i's vanish, independent of our perturbation. Now

$$e_3 = \binom{n}{3} - \left(\binom{n/2-1}{2} + \binom{n/2}{2}\right) \frac{n}{2} = \frac{n(n^2-4)}{24} \, ,$$

and the tightness of Theorem 11.18 is proved also for n even.

11.6 Characterizing ℓ-Vectors

A next step is to understand what possible ℓ-vectors for n points are, and to characterize and eventually count all possibilities for $\vec{\ell}$ and thus for $\vec{e}$. We start with two observations about $\vec{\ell}$.

Exercise 11.41. *Show that $\ell_{\lfloor (n-1)/2 \rfloor} \geqslant 1$. That is, there is always a bisecting edge.*

Exercise 11.42. *Show that if $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for all j, $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$.*

We summarize our knowledge about $\vec{\ell}$.

Theorem 11.43. *For $n \in \mathbb{N}$, the vector $\vec{\ell} = (\ell_0, \ldots, \ell_{n-1})$ of an n-point set satisfies the following conditions.*

- *All entries are nonnegative integers.*
- $\sum_{i=0}^{n-1} \ell_i = n$.
- $\ell_i = \ell_{n-1-i}$, *namely the entries are symmetric.*
- *If $\ell_i \geqslant 1$ for some $i \leqslant \lfloor (n-1)/2 \rfloor$, then $\ell_j \geqslant 1$ for $i \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. That is, starting from the first positive entry, the subsequent entries remain positive towards the middle.*

Let us call a vector of length n a *legal n-vector* if the conditions of Theorem 11.43 are satisfied. Then the only legal 1-vector is (1), the only legal 2-vector is $(1, 1)$, and the only legal 3-vectors are $(0, 3, 0)$ and $(1, 1, 1)$. The following scheme displays how we derive legal 6-vectors from legal 5-vectors, and how we can derive legal 7-vectors from legal 5- or 6-vectors.

$$
\overbrace{\begin{matrix} 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}}^{n=5}
\quad \xrightarrow{\text{add 1 to the middle and split}} \quad
\overbrace{\begin{matrix} 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}}^{n=6}
$$

$$
\text{add 2 to the middle } \downarrow \qquad\qquad\qquad\qquad \text{insert 1 in the middle } \downarrow
$$

$$
\underbrace{\begin{matrix} 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 0 \end{matrix}}_{n=7,\ \text{middle}>1}
\qquad\qquad
\underbrace{\begin{matrix} 0 & 0 & 3 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}}_{n=7,\ \text{middle}=1}
$$

Exercise 11.44. *Show that the scheme is complete when applied to odd* n*. That is, starting with all legal* n*-vectors,* n *odd, we can generate all legal* $(n+1)$*-vectors and all legal* $(n+2)$*-vectors this way.*

Exercise 11.45. *Show that the number of legal* n*-vectors is exactly* $2^{\lfloor (n-1)/2 \rfloor}$*.*

Exercise 11.46. *Show that every legal* n*-vector is the* ℓ*-vector of some set of* n *points in general position.*

With these exercises settled, we have given a complete characterization of all possible ℓ-vectors, thus of all possible e-vectors.

Theorem 11.47. *The number of different* e*-vectors (or* ℓ*-vectors) for* n *points is exactly* $2^{\lfloor (n-1)/2 \rfloor}$*.*

Exercise 11.48. *Show that* $\sum_{i=0}^{j} \ell_i \leqslant j+1$ *for all* $0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$*. (Hint: Otherwise, we get into conflict with the "remains positive towards the middle" property).*

11.7 More Vector Identities

We conclude the chapter with some additional identities that relate different vectors, many of which reveal illumimating combinatorial interpretations. The arguments are left as exercises. It is a good place for you to apply the mindset and methods from previous sections.

The first exercise gives an interpretation of the y_i's in relations (11.3).

Exercise 11.49. *For a set* P *of* n *points in general position, define a vector* $(b_0, \ldots, b_{n-2})$ *via the relations*

$$e_k = \binom{n}{k} - \sum_{i=0}^{n-2} \binom{i}{k-2} b_i = \sum_{i=0}^{n-2} \binom{i}{k-2} (n-i-1-b_i),$$

for $2 \leqslant k \leqslant n$*. Give a combinatorial interpretation of these numbers* b_i*.*

Next let us investigate how the vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ from relations (11.2), (11.3), and (11.4) connect to each other. Clearly, with e_1 and e_2 given, they determine each other. But how? This will allow us to relate the vectors $\vec{h}$ and $\vec{\ell}$.

Exercise 11.50. *Consider the relations (11.2)–(11.4) on* $\vec{x}_{0..n-3}, \vec{y}_{0..n-2}$*,* $\vec{z}_{0..n-1}$*, and* $\vec{e}_{1..n}$ *(using* $e_1 = e_2 = 0$*). Prove that the* y_i*'s are the forward differences of the* x_i*'s, and the* z_i*'s are the forward differences of the* y_i*'s. More concretely, show that*

$$y_i = \begin{cases} -x_0 & i = 0 \\ x_{i-1} - x_i & 1 \leqslant i \leqslant n-3 \\ x_{n-3} & i = n-2 \end{cases}$$

or equivalently, $y_i = x_{i-1} - x_i$ *for all* $0 \leqslant i \leqslant n-2$*, where* $x_{-1} := x_{n-2} := 0$*. Show that this entails* $x_i = -\sum_{j=0}^{i} y_j$ *for* $0 \leqslant i \leqslant n-3$*.*

Exercise 11.51. *Prove for vectors* $\vec{a}_{0..m}$ *and* $\vec{b}_{0..m}$*,*

$$a_k = \sum_{i=0}^{m} \binom{i}{k} b_i \quad \text{for all } 0 \leqslant k \leqslant m$$
$$\iff b_i = \sum_{k=0}^{m} (-1)^{i+k} \binom{k}{i} a_k \quad \text{for all } 0 \leqslant i \leqslant m.$$

Exercise 11.52. *Employing the previous exercise, what does* $h_0 = 1$ *say about* $\vec{e}_{3..n}$*?*

The following facts can now be readily derived.

Theorem 11.53.

$$h_i = \binom{i+2}{2} - \sum_{j=0}^{i} (i+1-j)\ell_j$$

Exercise 11.54. *Prove Theorem 11.53.*

Note that this implies the upper bounds we proved for the h_i's in Theorem 11.18, since $\sum_{j=0}^{i}(i+1-j)\ell_j$ is always nonnegative. Moreover, a combinatorial interpretation of the slack becomes evident.

Theorem 11.55.

$$e_k = \sum_{i=0}^{n} \binom{i}{k} (\ell_i - \ell_{i-1}) \quad \text{with } \ell_{-1} = \ell_n = 1$$

Exercise 11.56. *Prove Theorem 11.55.*

Let us point out some other counting problems that can be solved efficiently with the insights developed.

Exercise 11.57. *Given a ray* r *emanating from point* q*, and a point set* $P = \{p_1, \ldots, p_n\}$ *in the plane, design an efficient algorithm that counts the number of segments* $\overline{p_i p_j}$ *intersecting* r*. You may assume that* $P \cup \{q\}$ *is in general position and that* r *is disjoint from* P*.*

Exercise 11.58. *Let* w *be a line minus an interval on it (an infinite wall with a window). Given* n *points* P *in the plane, design an efficient algorithm that counts the number of pairs of points that can see each other, either because they are both on the same side of* w *or because they see each other through the window. You may assume general position.*

Exercise 11.59. *Recall that a point* $c \in \mathbb{R}^2$ *is a centerpoint of* $P \subset \mathbb{R}^2$ *if every halfplane containing* c *contains at least* $|P|/3$ *points from* P*. Identify the properties of* $\vec{e}$*,* $\vec{h}$ *and* $\vec{\ell}$ *which can certify that* 0 *is a centerpoint of* P*.*

Exercise 11.60. *Show that* $y_i = -y_{n-2-i}$ *and* $y_i \leqslant 0$ *for all* $0 \leqslant i \leqslant \lfloor \frac{n-2}{2} \rfloor$*. We refer here to the* y_i*'s as defined by (11.3). (Hint: You may wish to use Exercises 11.48 and 11.50.)*

Exercise 11.61. *Show that* $h_i \geqslant h_{i-1}$ *for all* $0 \leqslant \lfloor \frac{n-3}{2} \rfloor$*.*

Questions

64. Explain how the h-vector of a planar point set is defined via a lifting. Give the relation between the e-vector (number of embracing k-sets) and the h-vector.
65. Argue why the h-vector is independent of the lifting.
66. Argue why the h-vector is symmetric.
67. Argue why for a given generic lifting $P' \subset \mathbb{R}^3$ of a point set $P \subset \mathbb{R}^2$ in general position, removing a point cannot increase h_j, namely $h_j(P \setminus \{p\}) \leqslant h_j(P)$ for all $j \in \mathbb{N}_0$ and all $p \in P$.
68. Show how the ℓ-vector can be computed in $O(n \log n)$ time.
69. Argue why the ℓ-vector is symmetric ($\ell_i = \ell_{n-1-i}$ for all $0 \leqslant i \leqslant n-1$).
70. Explain orthogonal duals (Gale Duality). How do embracing sets and faces of polytopes relate to each other?