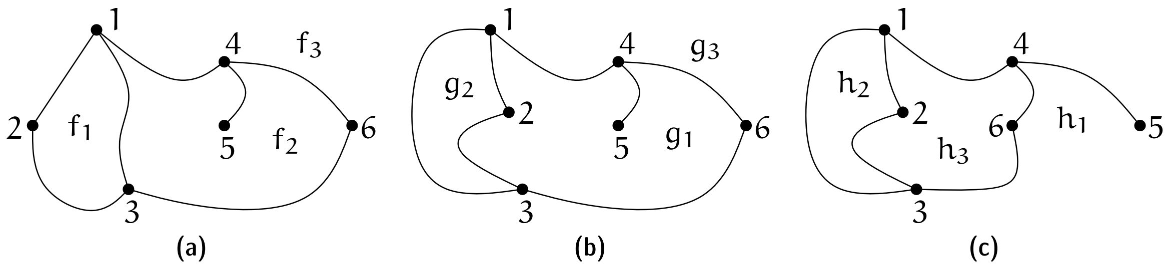


Figure 2.12: *Equivalent embeddings?*

(a) $\{(1,4,5,6,3),(1,3,6,2),(1,2,6,7,8,9,7,6,5),(7,9,8),(1,5,4)\}$

(b) $\{(1,4,5,6,3),(1,3,6,2),(1,2,6,7,8,9,7,6,5),(7,9,8),(1,4,5)\}$

Combinatorial embeddings are not only used to categorize plane graphs. They also play a role in algorithm design. Quite often, algorithms dealing with planar graphs do not need a full-fledged embedding to proceed. It is sufficient to operate on a combinatorial embedding, which is more efficient to handle.

Many people prefer a dual representation which, instead of listing face boundaries, enumerates the neighbors of v in cyclic order for each vertex v. It can avoid the issue of a vertex appearing multiple times in the sequence. However, the following lemma shows that such an issue does not arise when dealing with biconnected graphs.

Lemma 2.20. *In a biconnected plane graph every face is bounded by a cycle.*

We leave the proof as an exercise. Intuitively the statement is clear, but we believe it is instructive to think about a formal argument. An easy consequence is stated below, whose proof is also an exercise.

Corollary 2.21. *For any vertex v in a 3-connected plane graph, there is a cycle that contains all neighbours of v.*

Exercise 2.22. *Prove Lemma 2.20 and Corollary 2.21.*

Given Lemma 2.20, one might wonder the converse question: Which cycles in a planar graph G bound a face (in some plane embedding of G)? Such cycles are said to be *facial*; see Figure 2.13.

Exercise 2.23. *Describe a linear time algorithm that, given an abstract planar graph G and a cycle C in G, tests whether C is a facial cycle. (You may assume that planarity can be tested in linear time.)*

2.3 Unique Embeddings

As we have seen, an abstract planar graph may admit many different embeddings, even in the combinatorial sense. Under what condition does it admit a unique combinatorial embedding?

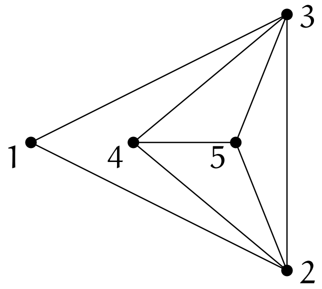


Figure 2.13: *The cycles* $(2,3,5)$ *and* $(1,2,5,3)$*, for example, are both facial. One can show that* $(2,4,3,5)$ *is not.*

To answer the question, we start by studying cycles that bound a face in *every* plane embedding of G. (Note that this is stronger than being facial.) The lemma below provides a complete characterization of these cycles. Let us agree on some terminology about a cycle C in a graph G. A *chord* of C is an edge in $E(G) \setminus E(C)$ that connects two vertices of C. The cycle C is *induced* if it does not have any chord. It is *separating* if $G \setminus C$ is not connected.

Lemma 2.24. *Let* G *be a planar graph which is neither a cycle, nor a cycle plus a single chord. Then a cycle* C *in* G *bounds a face in every plane embedding of* G *if and only if* C *is induced and not separating.*

Proof. "⇐": Consider any plane embedding Γ of G. By the Jordan Curve Theorem, the cycle C splits the plane into an interior and an exterior region. As $G \setminus C$ is connected, it lies either entirely in the interior or entirely in the exterior. In either case, the other region is bounded by C because C does not have any chord.

"⇒": Using contraposition, suppose that (1) C is not induced or (2) C is separating. We aim to find a plane embedding of G in which C does not bound a face. To this end, let us start from an arbitrary plane embedding Γ of G. If C does not bound a face in Γ then we are done. So next we assume that C bounds a face in Γ.

(1) If C is not induced, then it has a chord c. As $G \neq C \cup c$, the graph G either has some vertex $v \notin C$ or another chord $d \neq c$ of C. We modify Γ by rerouting the chord c inside the face C and obtain an embedding in which C does not bound a face: one of the two regions split by the Jordan curve C contains the chord c, and the other contains either the vertex v or another chord d.

(2) If C is separating, then $G \setminus C$ is not connected. If $G \setminus C = \emptyset$ then G is either C (which is excluded by assumption) or C plus some chords (which is handled by Case (1)). So from now on we assume $G \setminus C \neq \emptyset$ has two components A and B; see Figure 2.14a. Γ induces plane embeddings Γ_A of $A \cup C$ and Γ_B of $B \cup C$; the cycle C bounds a face in both of them. By the transformation in Theorem 2.2 we can make C bounding the outer face in Γ_A yet an inner face in Γ_B. Then we can glue the two embeddings at C, that is, extend Γ_B by adding Γ_A within the (inner) face

bounded by C (Figure 2.14b). The result is a plane embedding of G in which C does not bound a face.

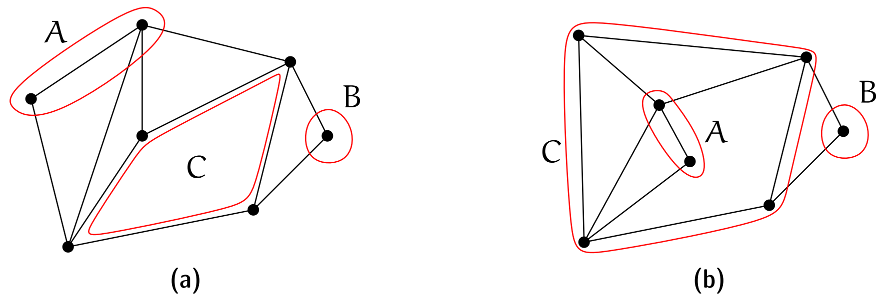


Figure 2.14: *A plane embedding in which* C *does not bound a face, in Case (2).*

□

For those special graphs G excluded in Lemma 2.24, it is easy to see that all cycles in G bound a face in every plane embedding. This completes the characterization. Since these special graphs are not 3-connected, we have

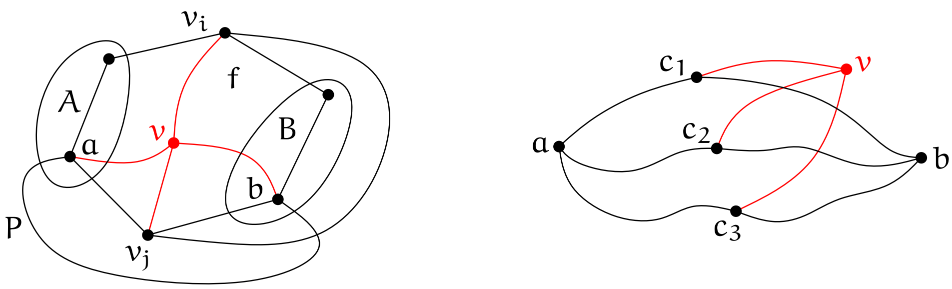
Corollary 2.25. *A cycle* C *of a 3-connected planar graph* G *bounds a face in every plane embedding of* G *if and only if* C *is induced and not separating.* □

The following theorem tells us that a wide range of graphs have little choice when embedded into the plane, from a combinatorial point of view. Geometrically, though, there is still much freedom.

Theorem 2.26 (Whitney [36]). *A 3-connected planar graph has a unique combinatorial plane embedding (up to equivalence).*

Proof. Let G be a 3-connected planar graph and suppose there exist two embeddings Φ_1 and Φ_2 of G that are not equivalent. So there is a cycle $C = (v_1, \ldots, v_k)$ in G that, say, bounds a face f in Φ_1 but does not bound any face in Φ_2. By Corollary 2.25 there are only two options:

Case 1: C has a chord $\{v_i, v_j\}$. Denote $A = \{v_x : i < x < j\}$ and $B = \{v_x : x < i \vee j < x\}$ and observe that both A and B are nonempty because $\{v_i, v_j\}$ is a chord and so v_i and v_j are not adjacent in C. Given that G is 3-connected, there is at least one path P from A to B that avoids both v_i and v_j. Let a denote the last vertex of P that is in A, and let b denote the first vertex of P that is in B. As C bounds f in Φ_1, we can add a new vertex v inside f and connect it to each of v_i, v_j, a and b by four pairwise internally disjoint curves. The result would be a plane graph that contains a K_5 subdivision with branch vertices v, v_i, v_j, a, and b. This contradicts Kuratowski's Theorem (Theorem 2.10).



(a) Case 1. **(b)** Case 2.

Figure 2.15: *Illustration of the two cases in Theorem 2.26.*

Case 2: C is induced and separating. Since C is induced and G is 3-connected, we must have $G \setminus C \neq \emptyset$. So $G \setminus C$ contains two distinct components A and B. Choose vertices $a \in A$ and $b \in B$ arbitrarily. Applying Menger's Theorem (Theorem 1.5) on the 3-connected graph G, there exist three paths $\alpha_1, \alpha_2, \alpha_3$, pairwise internally vertex-disjoint, from a to b. Let c_i be some vertex where α_i intersects C, for $1 \leqslant i \leqslant 3$. Note that c_1, c_2, c_3 exist because C separates A and B, and they are pairwise distinct because $\alpha_1, \alpha_2, \alpha_3$ are pairwise internally (vertex-)disjoint. Therefore, $\{a, b\}$ and $\{c_1, c_2, c_3\}$ form branch vertices of a $K_{2,3}$ subdivision in G. We can add a new vertex v inside f and connect it to each of c_1, c_2 and c_3 by three pairwise internally disjoint curves. The result would be a plane graph that contains a $K_{3,3}$ subdivision. This contradicts Kuratowski's Theorem (Theorem 2.10).

In both cases we arrived at a contradiction and so there does not exist such a cycle C. Thus Φ_1 and Φ_2 are equivalent. □

Whitney's Theorem does not provide a characterization of unique embeddability in general, as there are biconnected graphs with unique combinatorial plane embedding (such as cycles) as well as those with several, non-equivalent combinatorial plane embeddings (such as a triangulated pentagon).

Exercise 2.27. *Describe a family of biconnected planar graphs with exponentially many combinatorial plane embeddings. That is, show that there exists a constant* $c \in \mathbb{R}$ *such that for every* $n \in \mathbb{N}$ *there exists a biconnected planar graph on* n *vertices that has at least* c^n *different combinatorial plane embeddings.*

2.4 Triangulating a Planar Graph

We like to study worst case scenarios not so much to dwell on "how bad things could get" but rather—phrased positively—because worst case examples provide universal bounds of the form "things are always at least this good". Most questions related to embeddings get harder when the graph contains more edges because every additional edge poses an increasing danger of crossing. So let us study the worst case: planar graphs such that adding any edge shall break its planarity. These graphs are called *maximal planar*. Corollary 2.5 tells us that every (hence also maximal) planar graph on n vertices has at most $3n - 6$ edges. Yet we would like to learn a bit more about how these graphs look like.

Lemma 2.28. *A maximal planar graph on $n \geqslant 3$ vertices is biconnected.*

Proof. Consider a maximal planar graph $G = (V, E)$. Note that G is connected because adding an edge between two distinct components of a planar graph maintains planarity. Now if G is not biconnected, then it has a cut-vertex v. Take a plane drawing Γ of G. As $G \setminus v$ is disconnected, removal of v also splits $N_G(v)$ into at least two components. Hence there are two vertices $a, b \in N_G(v)$, consecutive in the circular order around v in Γ, that are in different components of $G \setminus v$. In particular, $ab \notin E$ and we can add this edge to G (routing it very close to the path (a, v, b) in Γ) without violating planarity. This is in contradiction to G being maximal planar, so G must be biconnected. □

Lemma 2.29. *In any embedding of a maximal planar graph on $n \geqslant 3$ vertices, all faces are topological triangles, that is, every face is bounded by exactly three edges.*

Proof. Consider a maximal planar graph $G = (V, E)$ and a plane drawing Γ of G. By Lemma 2.28 we know that G is biconnected and so by Lemma 2.20 every face of Γ is bounded by a cycle. Suppose that there is a face f in Γ bounded by a cycle $(v_0, \ldots, v_{k-1}, v_k = v_0)$ of $k \geqslant 4$ vertices. We claim that at least one of the edges v_0v_2 or v_1v_3 is not in E.

Suppose to the contrary that $\{v_0v_2, v_1v_3\} \subseteq E$. Then we can add a new vertex v' in the interior of f and connect it to each of v_0, v_1, v_2, v_3 by a curve inside f without introducing a crossing. In other words, given G is planar, the graph $G' = (V \cup \{v'\}, E \cup \{v'v_i : i \in \{0, 1, 2, 3\}\})$ is also planar. However, v_0, v_1, v_2, v_3, v' are branch vertices of a K_5 subdivision in G': v' is connected to all other vertices within f, each vertex v_i is connected to both $v_{(i-1) \bmod 4}$ and $v_{(i+1) \bmod 4}$ along the boundary of f, and the two missing connections are provided by the edges v_0v_2 and v_1v_3 (Figure 2.16a). This contradicts Kuratowski's Theorem. Therefore, one of the edges v_0v_2 or v_1v_3 must be absent from E, as claimed.

So assume without loss of generality that $v_1v_3 \notin E$. But then we can route a curve from v_1 to v_3 inside f in Γ without introducing a crossing (Figure 2.16b). It follows that the edge v_1v_3 can be added to G without sacrificing planarity, which is in contradiction to G being maximal planar. Therefore, there is no such face f bounded by four or more vertices. □

Theorem 2.30. *A maximal planar graph on $n \geqslant 4$ vertices is 3-connected.*

Exercise 2.31. *Prove Theorem 2.30.*

Exercise 2.32. *(a) A* minimal nonplanar *graph is a non-planar graph G which contains an edge e such that $G \setminus e$ is planar. Prove or disprove: Every minimal nonplanar graph contain an edge e such that $G \setminus e$ is maximal planar.*

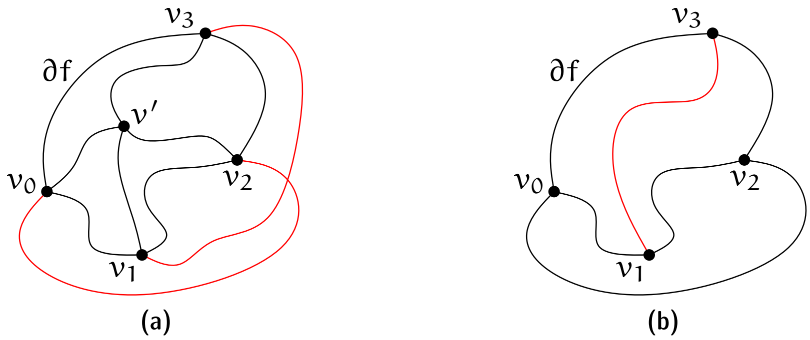


Figure 2.16: *Every face of a maximal planar graph is a topological triangle.*

(b) A maximal-plus-one planar *graph is a graph* G *that contains an edge* e *such that* $G \setminus e$ *is maximal planar. Prove or disprove: Every maximal-plus-one planar graph can be drawn with at most one crossing.*

Many questions about graphs are formulated only for connected graphs because it is easy to add edges to disconnected graphs and make them connected. For similar reason, many questions about planar embeddings are formulated only for maximal planar graphs because it is easy to augment planar graphs and make them maximal planar. Well, this last statement is not entirely obvious. Let us look at it in more detail.

An augmentation of a given planar graph $G = (V, E)$ to a maximal planar graph $G' = (V, E')$ where $E' \supseteq E$ is also called a *topological triangulation*. The proof of Lemma 2.29 already contains the basic algorithmic idea to topologically triangulate a plane graph.

Theorem 2.33. *For a given connected plane graph* $G = (V, E)$ *on* n *vertices one can compute in* $O(n)$ *time and space a maximal plane graph* $G' = (V, E')$ *with* $E \subseteq E'$.

Proof. Suppose, for instance, that G is represented as a DCEL[2], from which one can easily extract the face boundaries. As a clean-up, we walk along the boundary of each face. Whenever we see a vertex twice (or more), it must be a cut vertex. We fix this by adding an edge between its current predecessor and successor along the walk, and then continue the walk. Since the total number of traversed edges and vertices of all faces is proportional to $|E|$, which by Corollary 2.5 is linear, the clean-up finishes in $O(n)$ time. Henceforth we may suppose that all faces of G are bounded by cycles.

Every face that is bounded by more than three vertices selects an arbitrary vertex on its boundary. Conversely, every vertex keeps a list of all faces that have selected it. Then we process every vertex $v \in V$ as follows:

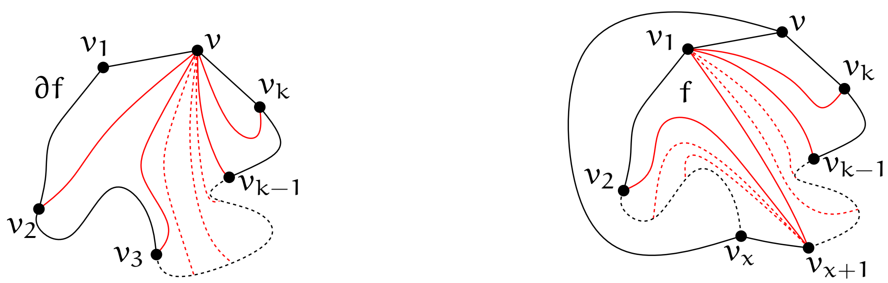
1. Mark all neighbors of v.

[2]If you wonder how the possibly complicated curves are represented: they do not need to be, since here we need a representation of the combinatorial embedding only.

2. For each face f that selected v, scan its boundary $\partial f = (v, v_1, \dots, v_k)$ counterclockwise, where $k \geqslant 3$, and find the first marked vertex $v_x \notin \{v_1, v_k\}$.

   - If there is no such vertex, we can safely triangulate f using a star from v, that is, by adding the edges vv_i, for $i \in \{2, \dots, k-1\}$ (Figure 2.17a). We then mark the new neighbors of v accordingly.
   - Otherwise, the edge vv_x as a curve embedded outside f prevents any vertex in $\{v_1, \dots, v_{x-1}\}$ from connecting to any vertex in $\{v_{x+1}, \dots, v_k\}$ by an edge in G. (The reasoning copies the one we made for the edges v_0v_2 and v_1v_3 in the proof of Lemma 2.29 above; see Figure 2.16a.) So we can safely triangulate f using a bi-star from v_1 and v_{x+1}, that is, by adding the edges v_1v_i, for $i \in \{x+1, \dots, k\}$, and v_jv_{x+1}, for $j \in \{2, \dots, x-1\}$ (Figure 2.17b).

3. After finishing all faces that seleted v, we conclude the processing of v by clearing all marks on its neighbors.



(a) Case 1: v does not have any neighbor on ∂f other than v_1 and v_k.

(b) Case 2: v has a neighbor v_x on ∂f other than v_1 and v_k.

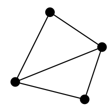
Figure 2.17: *Topologically triangulating a plane graph.*

Regarding the runtime bound, note that every face is visited only twice: one time when selecting its representative vertex, the other time when scanning its boundary. In this way, each edge is touched a constant number of times in step 2 overall. The marking/unmarking (steps 1 and 3) cost $\sum_{v \in V} \deg(v) = 2|E|$ time by the Handshaking Lemma. Therefore, the total time can be bounded by $O(n + |F| + |E|) = O(n)$ by Corollary 2.5. □

Using any of the standard planarity testing algorithms we can obtain a combinatorial embedding of a planar graph in linear time. Together with Theorem 2.33 this yields:

Corollary 2.34. *For a given planar graph $G = (V, E)$ on n vertices one can compute in $O(n)$ time and space a maximal planar graph $G' = (V, E')$ with $E \subseteq E'$.* □

The results discussed in this section can serve as a tool to fix the combinatorial embedding for a given graph G: augment G using Theorem 2.33 to a maximal planar graph G', whose combinatorial embedding is unique by Theorem 2.26.



Being maximal planar is a property of an abstract graph. In contrast, a geometric graph to which no straight-line edge can be added without crossing is called a *triangulation*. Not every triangulation is maximal planar, as the example depicted to the right shows.

It is also possible to triangulate a geometric graph in linear time. But this problem is much more involved. Triangulating a single face of a geometric graph amounts to what is called "triangulating a simple polygon". This can be done in near-linear[3] time using standard techniques, and in linear time using Chazelle's famous algorithm, whose description spans a fourty pages paper [9].

Exercise 2.35. *We discussed the DCEL structure to represent plane graphs in Section 2.2.1. An alternative way to represent an embedding of a maximal planar graph is the following: For each triangle, store pointers to its three vertices and to its three neighboring triangles. Compare both approaches. Discuss different scenarios where you would prefer one over the other. In particular, analyze the space requirements of both.*

Connectivity serves as an important indicator for properties of planar graphs. Already Wagner showed that a 4-connected graph is planar if and only if it does not contain K_5 as a minor. That is, assuming 4-connectivity the second forbidden minor $K_{3,3}$ becomes "irrelevant". For subdivisions this is a different story. Independently Kelmans and Semour conjectured in the 1970s that 5-connectivity allows to consider K_5 subdivisions only. This conjecture was proven only recently[4] by Dawei He, Yan Wang, and Xingxing Yu.

Theorem 2.36 (He, Wang, and Yu [18]). *Every* 5*-connected nonplanar graph contains a subdivision of* K_5*.*

Exercise 2.37. *Give a* 4*-connected nonplanar graph that does not contain a subdivision of* K_5*.*

Another example that illustrates the importance of connectivity is the following famous theorem of Tutte that provides a sufficient condition for Hamiltonicity.

Theorem 2.38 (Tutte [32]). *Every* 4*-connected planar graph is Hamiltonian.*

Moreover, for a given 4-connected planar graph a Hamiltonian cycle can also be computed in linear time [10].

2.5 Compact Straight-Line Drawings

As a next step we consider geometric plane embeddings, where every edge is drawn as a straight-line segment. A classical theorem of Wagner and Fáry states that this is not a restriction to plane embeddability.

[3] $O(n \log n)$ or—using more elaborate tools—$O(n \log^* n)$ time.

[4] The result was announced in 2015 and published in 2020.