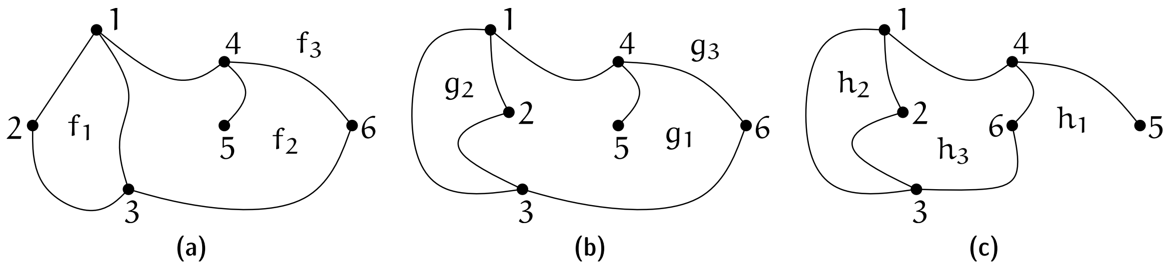


Figure 2.12: *Equivalent embeddings?*

(a) $\{(1,4,5,6,3),(1,3,6,2),(1,2,6,7,8,9,7,6,5),(7,9,8),(1,5,4)\}$

(b) $\{(1,4,5,6,3),(1,3,6,2),(1,2,6,7,8,9,7,6,5),(7,9,8),(1,4,5)\}$

Combinatorial embeddings are not only used to categorize plane graphs. They also play a role in algorithm design. Quite often, algorithms dealing with planar graphs do not need a full-fledged embedding to proceed. It is sufficient to operate on a combinatorial embedding, which is more efficient to handle.

Many people prefer a dual representation which, instead of listing face boundaries, enumerates the neighbors of v in cyclic order for each vertex v. It can avoid the issue of a vertex appearing multiple times in the sequence. However, the following lemma shows that such an issue does not arise when dealing with biconnected graphs.

Lemma 2.20. *In a biconnected plane graph every face is bounded by a cycle.*

We leave the proof as an exercise. Intuitively the statement is clear, but we believe it is instructive to think about a formal argument. An easy consequence is stated below, whose proof is also an exercise.

Corollary 2.21. *For any vertex v in a 3-connected plane graph, there is a cycle that contains all neighbours of v.*

Exercise 2.22. *Prove Lemma 2.20 and Corollary 2.21.*

Given Lemma 2.20, one might wonder the converse question: Which cycles in a planar graph G bound a face (in some plane embedding of G)? Such cycles are said to be *facial*; see Figure 2.13.

Exercise 2.23. *Describe a linear time algorithm that, given an abstract planar graph G and a cycle C in G, tests whether C is a facial cycle. (You may assume that planarity can be tested in linear time.)*

2.3 Unique Embeddings

As we have seen, an abstract planar graph may admit many different embeddings, even in the combinatorial sense. Under what condition does it admit a unique combinatorial embedding?

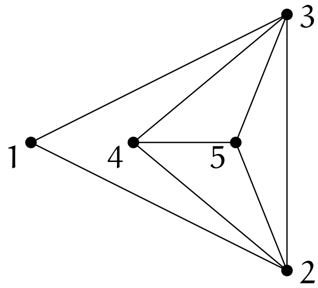


Figure 2.13: *The cycles* $(2,3,5)$ *and* $(1,2,5,3)$*, for example, are both facial. One can show that* $(2,4,3,5)$ *is not.*

To answer the question, we start by studying cycles that bound a face in *every* plane embedding of G. (Note that this is stronger than being facial.) The lemma below provides a complete characterization of these cycles. Let us agree on some terminology about a cycle C in a graph G. A *chord* of C is an edge in $E(G) \setminus E(C)$ that connects two vertices of C. The cycle C is *induced* if it does not have any chord. It is *separating* if $G \setminus C$ is not connected.

Lemma 2.24. *Let* G *be a planar graph which is neither a cycle, nor a cycle plus a single chord. Then a cycle* C *in* G *bounds a face in every plane embedding of* G *if and only if* C *is induced and not separating.*

Proof. "$\Leftarrow$": Consider any plane embedding Γ of G. By the Jordan Curve Theorem, the cycle C splits the plane into an interior and an exterior region. As $G \setminus C$ is connected, it lies either entirely in the interior or entirely in the exterior. In either case, the other region is bounded by C because C does not have any chord.

"$\Rightarrow$": Using contraposition, suppose that (1) C is not induced or (2) C is separating. We aim to find a plane embedding of G in which C does not bound a face. To this end, let us start from an arbitrary plane embedding Γ of G. If C does not bound a face in Γ then we are done. So next we assume that C bounds a face in Γ.

(1) If C is not induced, then it has a chord c. As $G \neq C \cup c$, the graph G either has some vertex $v \notin C$ or another chord $d \neq c$ of C. We modify Γ by rerouting the chord c inside the face C and obtain an embedding in which C does not bound a face: one of the two regions split by the Jordan curve C contains the chord c, and the other contains either the vertex v or another chord d.

(2) If C is separating, then $G \setminus C$ is not connected. If $G \setminus C = \emptyset$ then G is either C (which is excluded by assumption) or C plus some chords (which is handled by Case (1)). So from now on we assume $G \setminus C \neq \emptyset$ has two components A and B; see Figure 2.14a. Γ induces plane embeddings Γ_A of $A \cup C$ and Γ_B of $B \cup C$; the cycle C bounds a face in both of them. By the transformation in Theorem 2.2 we can make C bounding the outer face in Γ_A yet an inner face in Γ_B. Then we can glue the two embeddings at C, that is, extend Γ_B by adding Γ_A within the (inner) face

bounded by C (Figure 2.14b). The result is a plane embedding of G in which C does not bound a face.

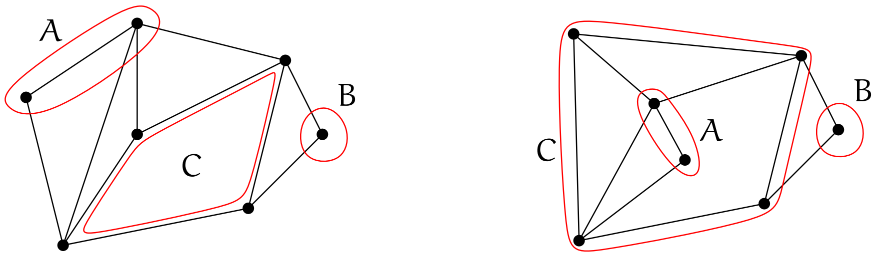


Figure 2.14: *A plane embedding in which C does not bound a face, in Case (2).*

□

For those special graphs G excluded in Lemma 2.24, it is easy to see that all cycles in G bound a face in every plane embedding. This completes the characterization. Since these special graphs are not 3-connected, we have

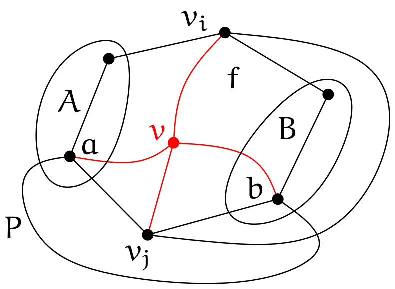
Corollary 2.25. *A cycle C of a 3-connected planar graph G bounds a face in every plane embedding of G if and only if C is induced and not separating.* □

The following theorem tells us that a wide range of graphs have little choice when embedded into the plane, from a combinatorial point of view. Geometrically, though, there is still much freedom.

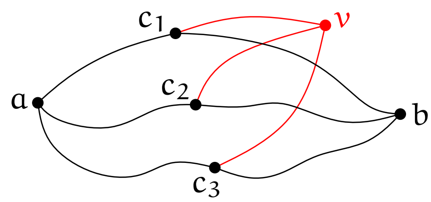
Theorem 2.26 (Whitney [36]). *A 3-connected planar graph has a unique combinatorial plane embedding (up to equivalence).*

Proof. Let G be a 3-connected planar graph and suppose there exist two embeddings Φ_1 and Φ_2 of G that are not equivalent. So there is a cycle $C = (v_1, \ldots, v_k)$ in G that, say, bounds a face f in Φ_1 but does not bound any face in Φ_2. By Corollary 2.25 there are only two options:

Case 1: C has a chord $\{v_i, v_j\}$. Denote $A = \{v_x : i < x < j\}$ and $B = \{v_x : x < i \vee j < x\}$ and observe that both A and B are nonempty because $\{v_i, v_j\}$ is a chord and so v_i and v_j are not adjacent in C. Given that G is 3-connected, there is at least one path P from A to B that avoids both v_i and v_j. Let a denote the last vertex of P that is in A, and let b denote the first vertex of P that is in B. As C bounds f in Φ_1, we can add a new vertex v inside f and connect it to each of v_i, v_j, a and b by four pairwise internally disjoint curves. The result would be a plane graph that contains a K_5 subdivision with branch vertices v, v_i, v_j, a, and b. This contradicts Kuratowski's Theorem (Theorem 2.10).



(a) Case 1.



(b) Case 2.

Figure 2.15: *Illustration of the two cases in Theorem 2.26.*

Case 2: C is induced and separating. Since C is induced and G is 3-connected, we must have $G \setminus C \neq \emptyset$. So $G \setminus C$ contains two distinct components A and B. Choose vertices $a \in A$ and $b \in B$ arbitrarily. Applying Menger's Theorem (Theorem 1.5) on the 3-connected graph G, there exist three paths $\alpha_1, \alpha_2, \alpha_3$, pairwise internally vertex-disjoint, from a to b. Let c_i be some vertex where α_i intersects C, for $1 \leqslant i \leqslant 3$. Note that c_1, c_2, c_3 exist because C separates A and B, and they are pairwise distinct because $\alpha_1, \alpha_2, \alpha_3$ are pairwise internally (vertex-)disjoint. Therefore, $\{a, b\}$ and $\{c_1, c_2, c_3\}$ form branch vertices of a $K_{2,3}$ subdivision in G. We can add a new vertex v inside f and connect it to each of c_1, c_2 and c_3 by three pairwise internally disjoint curves. The result would be a plane graph that contains a $K_{3,3}$ subdivision. This contradicts Kuratowski's Theorem (Theorem 2.10).

In both cases we arrived at a contradiction and so there does not exist such a cycle C. Thus Φ_1 and Φ_2 are equivalent. □

Whitney's Theorem does not provide a characterization of unique embeddability in general, as there are biconnected graphs with unique combinatorial plane embedding (such as cycles) as well as those with several, non-equivalent combinatorial plane embeddings (such as a triangulated pentagon).

Exercise 2.27. *Describe a family of biconnected planar graphs with exponentially many combinatorial plane embeddings. That is, show that there exists a constant $c \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ there exists a biconnected planar graph on n vertices that has at least c^n different combinatorial plane embeddings.*

2.4 Triangulating a Planar Graph

We like to study worst case scenarios not so much to dwell on "how bad things could get" but rather—phrased positively—because worst case examples provide universal bounds of the form "things are always at least this good". Most questions related to embeddings get harder when the graph contains more edges because every additional edge poses an