Chapter 3

Crossings

So far we have mostly studied planar graphs which allow us to avoid crossings altogether. However, there are many interesting graphs that are not planar, and still we would like to draw them in a reasonable fashion. An obvious quantitative approach is to minimize the number of crossings, even if they are inevitable.

3.1 Crossing Numbers

For an abstract graph $G = (V, E)$, the *crossing number* $\mathrm{cr}(G)$ is defined as the minimum number of edge crossings over all drawings of G. Analogously, the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ is defined as the minimum number of edge crossings over all straight-line drawings of G. A drawing of G that achieves $\mathrm{cr}(G)$ or $\overline{\mathrm{cr}}(G)$ crossings is called a *minimum-crossing drawing* or *minimum-crossing straight-line drawing*, respectively.

These notions are well-defined since $\mathrm{cr}(G) \leqslant \overline{\mathrm{cr}}(G) \leqslant \binom{|E|}{2}$ are finite. To see the upper bound, we construct a straight-line drawing of G as follows. Bijectively map the vertices of V onto a set of $n = |V|$ points in general position (that is, such that no three points are collinear), then draw every edge as a straight-line segment. This is a valid drawing in which every pair of distinct edges share at most one point.

Actually, this last property also holds for all minimum-crossing drawings, as the following lemma demonstrates.

Lemma 3.1. *In every minimum-crossing drawing of a graph, every pair of distinct edges share at most one point.*

Proof. Consider a graph G and a minimum-crossing drawing Γ of G. Suppose for contradiction that two edges $e \neq f$ share distinct points $p \neq q$ in Γ. Let e_p^q be the part of e from p to q in Γ, and similarly define f_p^q. Without loss of generality, suppose that e_p^q has no more crossings than f_p^q. Then we redraw f_p^q to closely follow e_p^q, as illustrated in Figure 3.1.

- If p (or q) is a common vertex of the edges e and f, then we can choose the side so that the crossing at q (or p) is eliminated.

- Otherwise, both p and q are crossing points. Depending on how f approaches p and q, we are able to eliminate either one (if approached from the same side of e) or two (if approached from opposite sides of e) of these crossings.

The number of crossings other than p and q does not increase, due to our assumption that e_p^q has at most as many crossings as f_p^q. Hence, the total number of crossings strictly decreases.

Finally, if f should cross itself due to this modification, we can eliminate each self-crossing by omitting the curve between the two occurrences along f. The result is a drawing of G with strictly fewer crossings than Γ, a contradiction to Γ being a minimum-crossing drawing of G. □

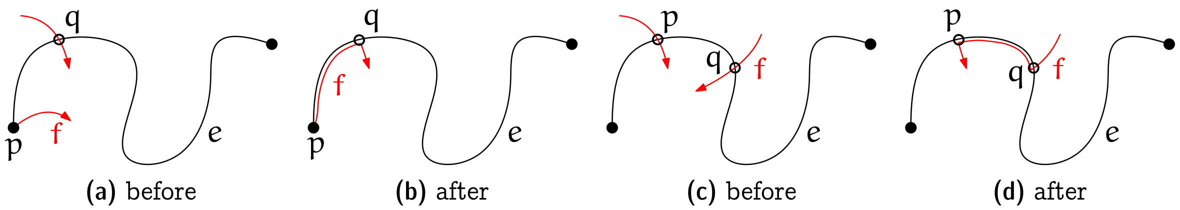

Figure 3.1: *Redraw f_p^q by the side of e_p^q to reduce the overall number of crossings. (a) and (b) depict the situation where both edges e and f are incident to a vertex p, in which case the crossing at q can be eliminated. (c) and (d) depict the situation where both p and q are crossings; in this example we can remove a crossing at p or q.*

A drawing in which every pair of edges has at most one point in common is called *simple*, and a graph drawn together with a simple drawing of it is called a *simple topological graph*. Using this terminology we can rephrase Lemma 3.1 as follows: "Every minimum-crossing drawing is simple." A simple drawing implies that no two adjacent edges cross. Drawings that satisfy this latter (and weaker) property are called *star-simple* because the incident edges to any vertex form a plane star.[1]

It is quite easy to certify an upper bound on the crossing number of a graph—just present a drawing that has a small number of crossings. But it is conceptually harder to certify a lower bound because it needs to account for *all* possible drawings of this graph. The following lower bound, though, can be obtained by simple counting.

Lemma 3.2. *For a graph G with $n \geqslant 3$ vertices and e edges, we have* $\mathrm{cr}(G) \geqslant e-(3n-6)$.

Proof. Consider a drawing of $G = (V, E)$ with $\mathrm{cr}(G)$ crossings. For each crossing, we pick one of the two involved edges arbitrarily. Obtain a new graph $G' = (V, E')$ from G by removing all picked edges. By construction G' is plane and, therefore, $|E'| \leqslant 3n - 6$ by Corollary 2.5. As at most $\mathrm{cr}(G)$ edges were picked ("at most" because some edge might

[1] In the literature also the terms *semi-simple* or *semisimple* are used.

be picked by several crossings), we have $|E'| \geqslant |E| - \mathrm{cr}(G)$. Combining both bounds completes the proof. □

Exercise 3.3. *Consider two edges* e *and* f *in a topological plane drawing so that* e *and* f *cross at least twice. Prove or disprove: There always exist two distinct crossings* p *and* q *of* e *and* f *so that the portion of* e *between* p *and* q *is not crossed by* f*, and the portion of* f *between* p *and* q *is not crossed by* e*.*

Exercise 3.4. *Let* G *be a graph with* $n \geqslant 3$ *vertices,* e *edges, and* $\mathrm{cr}(G) = e - (3n - 6)$*. Show that in every drawing of* G *with* $\mathrm{cr}(G)$ *crossings, every edge is crossed at most once.*

Exercise 3.5. *Consider the abstract graph* G *that is obtained as follows: Start from a plane embedding of the 3-dimensional cube, and add in every face a pair of (crossing) diagonals. Show that* $\mathrm{cr}(G) = 6 < \overline{\mathrm{cr}}(G)$*.*

Exercise 3.6. *A graph is* 1-planar *if it can be drawn in the plane so that every edge is crossed at most once. Show that a 1-planar graph* G *on* $n \geqslant 3$ *vertices has at most* $4n - 8$ *edges and* $\mathrm{cr}(G) \leqslant n - 2$*.*

3.2 The Crossing Lemma

The bound in Lemma 3.2 is quite good if the number of edges is close to $3n$ but not so good for dense graphs. For instance, for the complete graph K_n the lemma guarantees a quadratic number of crossings, whereas the Guy-Harary-Hill Conjecture [8] claims

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor \in \Theta(n^4).$$

The conjecture has been verified, in part with extensive computer help, for the complete graph on $n \leqslant 14$ vertices [2, 9, 11]; it remains open for $n \geqslant 15$.

So for dense graphs we would like to have a better lower bound for the crossing number. Given that the bound in Lemma 3.2 is reasonably good for sparse graphs, why not apply it to some sparse subgraph of G and then try scaling back to G? This simple idea turns out to work astonishingly well, as the following theorem demonstrates.

Theorem 3.7 (Crossing Lemma [4]). *For a graph* G *with* n *vertices and* $e \geqslant 4n$ *edges, we have*

$$\mathrm{cr}(G) \geqslant \frac{e^3}{64n^2}.$$

Proof. Consider a minimum-crossing drawing Γ of G, with $\mathrm{cr}(G)$ crossings. We select each vertex independently with probability p (a suitable value for p will be determined later). By this process we obtain a random subset $U \subseteq V$, the corresponding induced subgraph $G[U]$, along with its induced drawing $\Gamma[U]$. Consider the following three random variables:

- $N = |U|$, the number of selected vertices, with $\mathbb{E}[N] = pn$;
- M, the number of edges in $G[U]$, with $\mathbb{E}[M] = p^2 e$; and
- C, the number of crossings in $\Gamma[U]$, with $\mathbb{E}[C] = p^4 \mathrm{cr}(G)$. (Here we use Lemma 3.1, which says that adjacent edges do not cross in the minimum-crossing drawing Γ.)

According to Lemma 3.2, these quantities satisfy $C \geqslant \mathrm{cr}(G[U]) \geqslant M - 3N$ under all outcomes of the random experiment. Taking expectations on both sides and using linearity of expectation yields $\mathbb{E}[C] \geqslant \mathbb{E}[M] - 3\mathbb{E}[N]$ and so $p^4 \mathrm{cr}(G) \geqslant p^2 e - 3pn$. Setting $p = 4n/e$ (which is $\leqslant 1$ due to the assumption $e \geqslant 4n$) gives

$$\mathrm{cr}(G) \geqslant \frac{e}{p^2} - 3\frac{n}{p^3} = \frac{e^3}{16n^2} - 3\frac{e^3}{64n^2} = \frac{e^3}{64n^2}. \qquad \square$$

The beautiful proof described above is attributed to Chazelle, Sharir, and Welzl and listed in "Proofs from THE BOOK" [3, Chapter 40], a collection inspired by Paul Erdős' belief in "a place where God keeps aesthetically perfect proofs". The original proof of the Crossing Lemma was more complicated and had a worse constant.

Asymptotically the bound in Theorem 3.7 is tight: Pach and Tóth [10] describe graphs with $n \ll e \ll n^2$ that have crossing number at most

$$\frac{16}{27\pi^2}\frac{e^3}{n^2} < \frac{1}{16.65}\frac{e^3}{n^2}.$$

Hence it is not possible to replace $1/64$ by $1/16.65$ in the statement of the theorem. However, the constant $1/64$ is not the best possible: Ackerman [1] showed that $1/64$ can be replaced by $1/29$, at the cost of requiring $e \geqslant 6.95n$. Very recently, Büngener and Kaufmann [5] further improved the constant to $1/27.48$, at the cost of requiring $e \geqslant 6.77n$.

Exercise 3.8. *Show that the bound from the Crossing Lemma is asymptotically tight: There exists a constant c so that for every $n, e \in \mathbb{N}$ with $e \leqslant \binom{n}{2}$ there is a graph with n vertices and e edges that admits a plane drawing with at most ce^3/n^2 crossings.*

Exercise 3.9. *A graph is* quasiplanar *if it can be drawn in the plane such that no three edges pairwise cross. Denote by $\mathrm{qp}(n)$ the maximum number of edges in a quasiplanar graph on n vertices. Show that $\mathrm{qp}(n) \in O(n^{3/2})$.*

3.3 Applications of the Crossing Lemma

In the remainder of this chapter, we will discuss several nontrivial bounds on the size of combinatorial structures that can be obtained by judicious application of the Crossing Lemma. These beautiful connections were observed by Székely [13]; their original proofs were different and more involved.

We say that a point and a geometric object (such as a line or a circle) are *incident* if the former lies on the latter.

Theorem 3.10 (Szemerédi-Trotter [14]). *The maximum number of incidences between* n *points and* m *lines in* $\mathbb{R}^2$ *is at most* $2^{5/3} \cdot n^{2/3}m^{2/3} + 4n + m$.

Proof. Let P denote the given set of n points, and let L denote the given set of m lines. We may suppose that every line from L contains at least one point from P. (Discard all lines that do not, as they contribute no incidence.) Denote by I the number of incidences between P and L. Consider the graph $G = (P, E)$ whose vertices are the points P, and where two points p, q are joined by an edge if they appear consecutively along some line $\ell \in L$ (that is, $p, q \in \ell$ and no other point from P lies on the line segment $\overline{pq}$). The arrangement of P and L naturally induces a straight-line drawing of G. It has at most $\binom{m}{2}$ crossings because every crossing must be an intersection of two lines, and any two lines can intersect at most once.

Each line $\ell \in L$ is incident to some $I_\ell \geqslant 1$ point(s) from P and contributes $I_\ell - 1$ edge(s) to E. Hence $|E| = \sum_{\ell \in L}(I_\ell - 1) = I - m$. If $|E| \leqslant 4n$, then $I \leqslant 4n + m$ and the theorem holds. Otherwise, we can apply the Crossing Lemma to obtain

$$\binom{m}{2} \geqslant \mathrm{cr}(G) \geqslant \frac{|E|^3}{64n^2} = \frac{(I-m)^3}{64n^2}$$

and so $I \leqslant 2^{5/3} n^{2/3} m^{2/3} + m$. □

The bound in Theorem 3.10 is asymptotically tight, in the following sense [10, Remark 4.2]. There exist sets of n points and m lines in $\mathbb{R}^2$ that have $c \cdot n^{2/3}m^{2/3}$ incidences, for some constant $c > 0.42$ that is independent of n and m.

Theorem 3.11. *The maximum number of unit distances between* n *points in* $\mathbb{R}^2$ *is at most* $5n^{4/3}$.

Proof. Let P be the given set of n points, and consider the set C of n unit circles centered at the points in P. Then the number I of incidences between P and C is exactly twice the number of unit distances between points from P. So it suffices to upper bound I.

Define a graph $G = (P, E)$ on P as follows. For each circle $c \in C$, we list the points from $P \cap c$ in circular order, and add a new edge between every pair of consecutive points. By construction, if c contains I_c points from P, then it contributes exactly I_c edges to E, hence $I = |E|$. Note however that G is not necessarily simple, as it may contain loops (if some $I_c = 1$) and parallel edges (if some $I_c = 2$, or if some $p, q \in P$ are consecutive along different circles).

Obtain a new graph $G' = (P, E')$ from G by removing all edges along circles $c \in C$ of $I_c \leqslant 2$. Since at most $|C| = n$ circles are removed and each removed circle contributed at most two edges to E, we have $|E'| \geqslant |E| - 2n$. In G' there are neither loops, nor parallel edges contributed by the same circle. Therefore, between any two points p and q there are at most two parallel edges in G' because at most two different unit circles can pass through any two points $p \neq q$ in $\mathbb{R}^2$.

Obtain a new graph $G'' = (P, E'')$ from G' by removing one copy of every double edge. Clearly G'' is a simple graph with $|E''| \geqslant |E'|/2 \geqslant |E|/2 - n$. Rearranging, we have $I = |E| \leqslant 2(|E''| + n)$.

If $|E''| \leqslant 4n$, then $I \leqslant 10n < 10n^{4/3}$ and the theorem holds. Otherwise, by the Crossing Lemma we have

$$n^2 > 2\binom{n}{2} \geqslant \operatorname{cr}(G'') \geqslant \frac{|E''|^3}{64n^2}.$$

Here the upper bound on $\operatorname{cr}(G'')$ is due to that every pair of circles can intersect at most twice. Rearranging, it follows that $|E''| < 4n^{4/3}$ and so $I < 8n^{4/3} + 2n < 10n^{4/3}$. □

Exercise 3.12. *Show that the maximum number of unit distances determined by* n *points in* $\mathbb{R}^2$ *is* $\Omega(n \log n)$. *Hint: Consider the hypercube.*

The final application comes from arithmetic combinatorics. Given a set $A \subset \mathbb{R}$, we denote the *sum set* by $A + A := \{a + a' : a, a' \in A\}$ and similarly the *product set* by $A \cdot A := \{a \cdot a' : a, a' \in A\}$. It is easy to construct ground sets that have a small, that is, linear size sum set: Just take an arithmetic progression, such as $2, 4, 6, 8, 10, \ldots$. Similarly, geometric progressions exhibit a small product set. However, it is much more challenging to find a ground set A for which both the sum set and the product set are small. In fact, Erdős conjectured [7] that for every set A of n numbers, we have $\max\{|A+A|, |A \cdot A|\} \in \Omega(n^{2-\epsilon})$, for every $\epsilon > 0$. The general conjecture is still open. But the statement is known to hold for *some* reasonably small values of ϵ. At a first glance, it is not so clear why there should be a connection between this problem and questions about crossings in drawings of graphs. But there is such a connection, as discovered by Elekes [6]. He used the Crossing Lemma to give an elegant proof of the following bound.

Theorem 3.13 (Elekes [6]). *For* $A \subset \mathbb{R}$ *with* $|A| = n \geqslant 3$ *we have*

$$\max\{|A + A|, |A \cdot A|\} \geqslant \frac{1}{4} n^{\frac{5}{4}}.$$

Proof. Let $A = \{a_1, \ldots, a_n\}$. Set $X = A + A$ and $Y = A \cdot A$. We will show that $|X||Y| \geqslant \frac{1}{16} n^{5/2}$, which proves the theorem. Let $P = X \times Y \subset \mathbb{R}^2$ be the set of points whose x-coordinate is in X and whose y-coordinate is in Y. So we have $|P| = |X||Y|$. Next define a set L of lines by $\ell_{ij} = \{(x, y) \in \mathbb{R}^2 : y = a_i(x - a_j)\}$, for $i, j \in \{1, \ldots, n\}$. Clearly, we have $|L| = n^2$.

On the one hand, every line ℓ_{ij} contains at least n points from P because for each $k \in \{1, \ldots, n\}$, the point $(x_k, y_k) := (a_j + a_k, a_i a_k) \in X \times Y$ satisfies the equation $y_k = a_i(x_k - a_j)$ and thus is on ℓ_{ij}. Therefore the number I of incidences between P and L is at least $|L| \cdot n = n^3$.

On the other hand, by the Szemerédi-Trotter Theorem we have

$$I \leqslant 2^{5/3} |P|^{2/3} n^{4/3} + 4|P| + n^2.$$

Combining both bounds we obtain

$$2^{5/3} |P|^{2/3} n^{4/3} + 4|P| + n^2 \geqslant n^3.$$

Hence, at least one of the two summands $2^{5/3}|P|^{2/3}n^{4/3}$ and $4|P| + n^2$ is at least half of the sum, that is, at least $n^3/2$. If it is the latter, then we have

$$|P| \geqslant \frac{n^2}{4}\left(\frac{n}{2} - 1\right).$$

Using that $n \geqslant 3$ and therefore $\sqrt{n} \geqslant 3/2$, we continue to bound

$$\frac{n^2}{4}\left(\frac{n}{2} - 1\right) = \frac{n^2}{4}\left(\frac{\sqrt{n}\sqrt{n}}{6} + \frac{n}{3} - 1\right) \geqslant \frac{n^2}{4}\frac{\sqrt{n}}{4} = \frac{n^{5/2}}{16}.$$

To conclude the proof it remains to consider the former case, in which

$$|P|^{2/3} \geqslant \frac{n^3}{2 \cdot 2^{5/3}\, n^{4/3}} = \left(\frac{n^5}{256}\right)^{1/3} \implies |P| \geqslant \frac{n^{5/2}}{16}.$$

□

The lower bound has been gradually improved in a series of papers. The current state of the art is

$$\max\{|A + A|, |A \cdot A|\} \geqslant n^{\frac{4}{3} + \frac{2}{1167}} > n^{1.335}$$

by Rudnev and Stevens [12].

Questions

8. *What is the crossing number of a graph? What is the rectilinear crossing number?* Give the definitions and examples. Explain the difference.
9. *For a nonplanar graph, the more edges it has, the more crossings we would expect. Can you quantify such a correspondence more precisely?* State and prove Lemma 3.2 and Theorem 3.7 (The Crossing Lemma).
10. *Why is it called "Crossing Lemma" rather than "Crossing Theorem"?* Explain at least two applications of the Crossing Lemma, for instance, your pick out of the Theorems 3.10, 3.11, and 3.13.

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