INFORMATIK
für Mathematiker und Physiker

Eine Einführung in C++

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[1]This book originates from a set of lecture notes written by Joachim Giesen and the authors in 2005.

Contents

Chapter 1

Introduction

1.1 Why learn programming?

You can tell I'm educated, I studied at the Sorbonne
Doctored in mathematics, I could have been a don
I can program a computer, choose the perfect time
If you've got the inclination, I have got the crime

Pet Shop Boys, Opportunities (1986)

This section explains what a computer program is, and why it is important for you not only to use *computer programs, but also to* write *them.*

When people apply for a job these days, their resume typically contains a section called *computer skills*. Items listed there might include *Word*, *Excel*, or *Powerpoint*. These are the names of *application programs*, programs that have been written by certain people (in the above cases, at Microsoft corporation) to be used by other people (for example, a sales representative).

The *computer skills* section might also list items like *HTML*, *Java*, or *C++*. These are the names of *programming languages*, languages used to instruct, or program, a computer. Using a programming language, *you* can write the programs that will subsequently be used by others, or by yourself.

A computer program is a list of instructions to be automatically processed by a computer. The computer itself is stupid—all the intelligence comes from the program. In this sense, a program for the computer is like a cookbook recipe for someone who cannot cook: even with very limited skills, impressive results can be obtained, through a step-by-step instruction.

Most people simply use programs, just like they use cookbooks. A sales representative, for example, needs application programs as tools for his work. The fact that you are reading this lets us believe that you potentially belong to the category of people who also need to write programs.

There are many reasons for writing programs. Some employer might pay for it, some bachelor course might require it, but ultimately, there is a deeper reason behind it that we plan to explain next. The upshot is that nowadays, you cannot be a serious engineer, let alone a serious scientist, without at least some basic programming skills. Even in less serious contexts, we can recommend to learn programming, because it can bring about a lot of fun and satisfaction.

In the twentieth century, computers have revolutionized the way science and engineering are done. To be more concrete, we will underpin this with an example from mathematics. You probably don't expect math to be mentioned first in connection with computers; indeed, many mathematicians still use paper and pencil on a daily basis. But *what* they write down has changed. Before computers were available, it was often necessary to write down actual numbers, and to perform calculations with them by hand. This happened not so much in writing proofs for new theorems, but in the process of *finding* these theorems. This process often requires to go over many concrete examples, or counterexamples, in order to see certain patterns, or to discover that some statement is false. The computer has tremendously accelerated this process by taking over the routine work. When you look at a mathematician's notepad today, you still find greek letters and all kinds of strange symbols, but most likely no numbers larger than ten.

There is one topic that nicely illustrates the situation, and this is the search for the *Mersenne primes*. In 1644, the French monk and mathematician Marin Mersenne established the following claim.

> **Mersenne's Conjecture.** *The numbers of the form* $2^n - 1$ *are prime numbers for* $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$*, but for no other number* $n < 257$.

Mersenne corresponded with many of the leading mathematicians at that time, so his conjecture became widely known. Up to $n = 7$, you can verify it while you read this, and in 1644, the conjecture was already verified up to $n = 19$.

It took more than hundred years until the next exponent on Mersenne's list could be verified. In a letter to Bernoulli published in 1772, Leonhard Euler proved that $2^{31} - 1 = 2147483647$ is a prime number. But in 1876, another hundred years later, Mersenne posthumously received a heavy blow. Edouard Lucas proved that $2^{67} - 1 = 147573952589676412927$ is *not* a prime number (Lucas showed his passion for large numbers also when he invented the *Tower of Hanoi* puzzle). Lucas's proof does not work the way you would expect: it does *not* exhibit a prime factor of $2^{67} - 1$ (the most direct way of proving that a number is not prime), but it uses a clever indirect argument invented by Lucas in the same year. The factorisation of $2^{67} - 1$ remained unknown for another 25 years.

In 1903, Frank Nelson Cole was scheduled to give a lecture to the American Mathematical Society, whose title was 'On the Factorisation of Large Numbers'. Cole went to the blackboard, and without saying a single word, he first wrote down a calculation to obtain $2^{67} - 1$ by repeated multiplication with two. He finally had the number

```
147573952589676412927
```

on the blackboard. Then he wrote down another (much more interesting) calculation for the product of two numbers.

```
761838257287 x 193707721
------------------------
    761838257287
    6856544315583
     2285514771861
      5332867801009
```

```
          5332867801009
           5332867801009
            1523676514574
              761838257287
     ---------------------
     147573952589676412927
```

Cole had proved that $2^{67}-1 = 761838257287 \cdot 193707721$, making the result of Lucas believable to everybody: $2^{67}-1$ is not a prime number! He received standing ovations for this accomplishment and later admitted that he had worked on finding these factors every Sunday for the last three years.

Today, you can start a computer algebra program on your computer (a popular one is `Maple`), type in

```
ifactor(2^67-1);
```

and within less than a second get the output

```
(761838257287)(193707721)
```

To summarize: hundred years ago, a brilliant mathematician needed three years to come up with a result that much less brilliant people (we are not talking about you) could get in less than a second today, using a computer and the right program. This seems disturbing at first sight, and thinking about the precious time of his life Cole devoted to the problem, you may even feel sorry for him. You shouldn't; rather, the story has three important lessons in store.

Tool skills. Lesson one is that Cole's calculations were extremely difficult, given the tools he had (paper, pencil, and probably very good mental arithmetic). Given the tools *you* have (the computer and a computer algebra program called `Maple`), Cole's calculations are easy routine. We are sure that Cole would feel sorry for anyone using these new tools only to reproduce some hundred-year old calculation. Useful new tools lead to new possibilities and challenges. On the one hand, this *allows* you to do more than you could do before; on the other hand it also *forces* you to do more if you want to keep up with the developments. Whatever you do, nowadays you must acquire and maintain at least some basic knowledge of computers and application programs.

Problem skills. Lesson two is that tool skills alone would not have helped Cole to factor $2^{67}-1$. Cole also was a good mathematician who knew a lot of theory he could use to save calculations. This is the reason why he "only" needed three years.

Even nowadays, computers and application programs are not everything. Factoring $2^{67}-1$ is easy because this is a small number by today's standards. But factoring large numbers (2^{1000} is considered large today; in a couple of years, it might be 2^{2000}) is still a very difficult problem for which no efficient solutions are known. The problem of factoring large numbers is the most prominent problem for which most people must actually hope that *no* efficient solution will ever be found. The reason is that many cryptosystems that are in use (think of secure internet banking) are purely based on the practical impossibility of factoring large numbers. Therefore, the worst scenario for the networked world would be that the "bad guys" discover first how to factor large numbers efficiently.

There are many other problems that are as far from a solution as they were in pre-computer days. Coming back to Mersenne, we still cannot characterize the exponents n for which the number $2^n - 1$ is a prime number. We don't even know whether there are infinitely many such Mersenne primes. If you plan to make a contribution here, you should not buy a faster computer with the latest version of `Maple`, but study math. Even in the case of problems for which computers can really contribute to (or actually find) the solution, you typically need to have a deep understanding of the problem in order to know *how* to use the computer. If you want to become an engineer or a scientist, you must acquire and maintain a profound knowledge about the problems you will be dealing with. This fact was true hundred years ago, and it is still true—computers have not yet learned to solve interesting problems by themselves.

Programming Skills. Lesson three is one that Cole did not live to see: nowadays, problem-specific knowledge can be turned into problem-specific computer programs. That way, the state of the art concerning Mersenne primes has advanced quite far. It turned out that Mersenne had made five mistakes: $n = 67$ and $n = 257$ in Mersenne's list do not lead to prime numbers; on the other hand, Mersenne had "forgotten" the exponents $n = 61, 89$ and 107.

As of September 2008, we know 46 Mersenne primes, the largest of which has an exponent of $n = 43, 112, 609$.[1] But don't believe that this one was found with off-the-shelf programs.

Problems occuring in the daily life of an engineer or a scientist are often not easy to solve, even with a computer and standard software at hand. In order to attack them, you need *tool skills* for the routine calculations, and *problem skills* to understand and extract the aspects of the problem that can in principal be solved by a computer. But in the end, you need *programming skills* to actually do it.

The art of computer programming. To conclude this section, let us be honest: for many people (including the authors of this book), the process of writing programs has some very non-utilitarian aspects as well. We have mentioned two of them before: fun and satisfaction. We could add mathematical beauty and ego boost. In one way or another, every passionate programmer feels at least a little bit like an artist.

The prime advocator of this view on programming is Donald E. Knuth. He is the author of a monumental and seminal series of seven books entitled *The Art of Computer Programming*. Starting with Volume I in 1968, three of the seven volumes are published

[1] see `www.mersenne.org`

by now. Drafts of Volume IV circulate since 2005, and the planned release date of Volume V is 2015 (it should be added that Knuth was born in 1938, and on his webpage `http://www-cs-faculty.stanford.edu/~knuth/taocp.html`, he at least implicitly mentions the possibiliy that Volumes VI and VII will not be written anymore).

Let Knuth have the final say here (a quote from the beginning of Volume I):

> *The process of preparing programs for a digital computer is especially attractive, not only because it can be economically and scientifically rewarding, but also because it can be an aesthetic experience much like composing poetry or music.*

1.2 How to run a program

> *In Paris they just simply opened their eyes and stared when we spoke to them in French! We never did succeed in making those idiots understand their own language.*
>
> *Mark Twain, The Innocents Abroad (1869)*

This section explains what it really means to "write a program", and how you enable the computer to run it. For this, we describe the ingredients involved in the process: the editor*, the* compiler*, the* computer *itself, and the* operating system*. Computer, compiler and operating system together form the* platform *on which you are writing programs.*

1.2.1 Editor

Writing a program is not so different from writing a letter. One composes a text, that is, a (hopefully) meaningful sequence of characters. Usually, there are certain conventions on how such a text is structured, and the purpose of the text is to transport information.

What has been said so far applies to both letters and programs. But when writing a program, there is another aspect that has to be taken into account: A program has to be "read" by a computer, meaning that it must be available to the computer in electronic form. In the future, we might be able to orally dictate the program to the computer, but nowadays, the common way is to use a keyboard and simply type it in. An *editor* is an application program that allows you to display, modify, and electronically store such typed-in text. The use of editors is not restricted to programming, of course. With some still existing romantic exceptions, even letters are composed using editors such as *Word*.

1.2.2 Compiler

Making a program available to the computer in electronic form is usually not enough. The *machine language* a computer can understand directly is very primitive and quite different from natural languages.

Writing the programs in machine language is no viable alternative, since that would require to break the program into a large number of primitive instructions that the computer can understand. This is like telling your friend to come over for dinner by telling her which muscles to move in order to get to your place.

Moreover, machine languages vary considerably between different computers. That is, in order to use a program written for one specific computer A on a different computer B, one first has to translate the program from the machine language of A to the

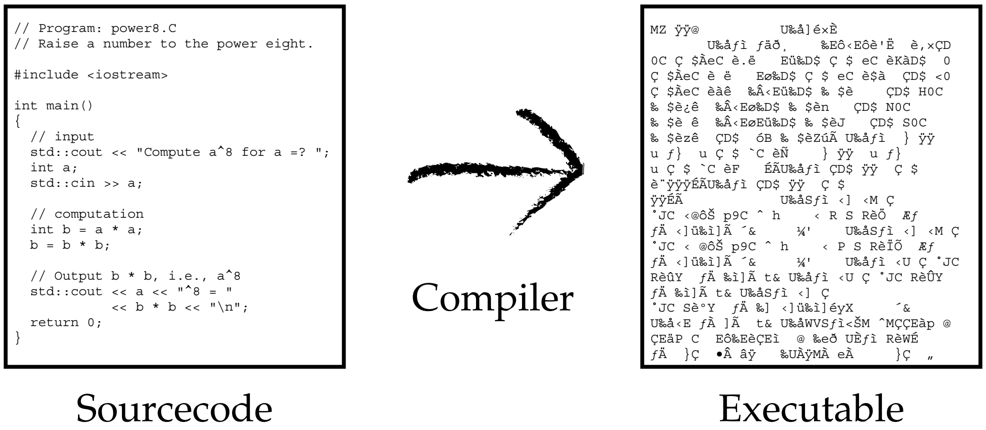


Figure 1: *A compiler translates the sourcecode into an executable program.*

machine language of B. This process, called *porting*, can be very cumbersome if the machine languages of A and B are substantially different. Also, porting can only be done with a detailed knowledge of the peculiarities of the involved computers. But this type of knowledge is not generally worthwhile to acquire, as it is tied to one very specific computer. As soon as this computer is replaced by another one, major parts of such computer-specific knowledge become worthless and have to be rebuilt from scratch.

To reduce this undesirable entanglement of computers and programs, and to allow us to write programs in less primitive language, (high-level) programming languages have been developed. These are standardized languages that form a kind of compromise between natural languages and machine language. Indeed, the use of the word "compromise" is justified because there are two conflicting goals: On the one hand, we would like to write programs in a language that is as close to natural language as possible. On the other hand, we have to make the computers understand the programming language as well; this task is obviously much easier if the programming language is close to machine language.

What does it mean "to make the computers understand the programming language"? In the end, any program has to be translated into machine language. The process of this translation is called *compilation*. Now you will probably ask: "Where is the benefit of this whole programming language concept? In order to do the translation I still have to know all these computer-specific details, don't I?" Right. If you would have to translate the program yourself. The key is: You are not supposed to translate it yourself. Instead, let a program do it for you. Such a program is referred to as a *compiler*; it translates a given program in a programming language, the *sourcecode*, into a program in machine language, the *executable*. See Figure 1 for an illustration.[2]

In summary: The big benefit of (high-level) programming languages is that they

[2]The picture of the executable is somewhat inappropriate, since it does not show what the computer gets to see after compilation, but rather what *you* might see when you (accidentally) load the executable into the editor. The main point that we are trying to make here is that the executable is not human-readable.

abstract from the capabilities of specific computers. Programs written in a high-level language can be run on all kinds of computers, as long as a compiler for the language is available on the particular computer.

1.2.3 Computer

If you are not interested in writing compilers, it is not necessary to understand in detail how a computer works. But there are some basic principles behind the design of most computers that *are* important to understand. These principles form the *von Neumann architecture*, and they are important, since almost all programming languages are tailored to the von Neumann architecture.

Any computer with von Neumann architecture has a *random access memory* (RAM, or simply main memory), and a *central processing unit* (CPU, or simply processor). The main memory stores the program to be run, but also data that the program requires as input, and data that the program produces as output. The processor is the "brain" of the computer: it executes the program, meaning that it carries out the sequence of instructions prescribed by the program in machine language.

Main memory. You can think of the computer's main memory as a long row of switches, each of them being either on or off. During program execution, switches are flipped. At any time, the memory content—the current positions of all switches—defines the *program state*. The program state completely determines what happens next. Conceptually, we also consider user input and program output as part of the program state, even though the corresponding "switches" might be in the user's brain, or on printed paper.

Since modern computers are capable of flipping several switches (32, say) at the same time, consecutive switches are accordingly grouped into *memory cells*. The positions of all switches in the cell define the *content* of the cell; in more abstract terms, the switches are called *bits*, each of them capable of storing one of the numbers $\{0, 1\}$. In this sense, you can interpret the content of a memory cell as a binary number with, for example, 32 digits. We also say that we have a 32-bit machine, or a *32-bit system*.

Each memory cell is uniquely identified by its *address*. You can think of the address simply as the position of the memory cell in the list of all memory cells.

To look up bit values, or to flip bits within a specific memory cell, the cell has to be *accessed* through its address. Think of a robot arm with 32 fingers that you can tell to move to memory cell number 17.

The term *random access* refers to a physical property of the computer's memory: the time it takes to access a cell (to "move to its bits") is the same for all cells; in particular, it does *not* depend on the address of the cell. When you think in terms of the robot arm analogy, it becomes clear that random access cannot be taken for granted. It is not necessary to discuss the physical means by which random access is realized; the important point here is that random access frees us from thinking about where to store a data item in order to access it efficiently.

Processor. You can think of the computer's processor as a box that is able to load and then execute the machine language instructions of a program in order. The processor has some memory cells of its own, called registers, and it can transfer data from the computer's main memory to its registers, and vice versa. The register contents are also part of the program state. Most importantly, the processor can perform a fixed set of simple operations (like adding or subtracting register contents), directly corresponding to the machine langauge instructions. This is where the functionality of the whole program comes from in the end. Even very complicated and useful programs can be put together from a simple set of machine language instructions.

A single instruction acts like a mathematical function: given the current program state, a valid instruction generates a new and well-defined next program state. This implies that any sequence of instructions, and in particular the whole program has a well-defined behavior, depending on the initial program state.

1.2.4 Operating system

We have seen that in order to write a program and run it, you first have to start an editor, type in the program, then call the compiler to translate the program into machine language, and finally tell the computer to execute it. In all this "starting", "calling" and "telling", you rely on the computer's *operating system* (OS), a program so basic that you may not even perceive it as a program. Popular operating systems are *Windows*, *Unix*, *Linux*, and *Mac OS*.

For example, whether you start the editor by clicking on some icon, or whether you type a command for this somewhere, the operating system makes sure that the editor program is loaded into the main memory, and that the processor starts executing it. Similarly, when you store your written program, the operating system allocates space for it on the hard disk and associates it with the file name you have provided.

A computer without operating system is like a car without tires, and most computers you can buy come with a pre-installed operating system. It is important to understand, though, that the operating system is not inextricably tied to the computer: you can take your "Windows PC" and reinstall it under Linux.

1.2.5 Platform

The computer, its operating system and the compiler are together referred to as the *platform* on which you are writing your programs. The editor is not part of the platform, since it does not influence the behavior of the program.

In an ideal world, there is no need for you to know the platform when you are writing programs in a high-level programming language. Recall that the plan is to delegate the platform-specific aspects to the compiler. A typical such platform-specific aspect is the size of a memory cell, i.e. the number of bits that can be manipulated together. This is mostly 32 these days, but for some computers it is 64, and for very primitive computers (like they are used in smart cards, say), it can be much less than 32.

When you are using or relying on machine-oriented features of the programming language, platform-specific behavior might be the result. Many high-level programming languages have such low-level features to facilitate the translation into *efficient* machine language.

Your goal should always be to write *platform-independent* code, since otherwise, it may be very difficult to get your program to run on another computer, even if you have a compiler for that computer. This implies that certain features should be avoided, even though it might seem advantageous to use them on a specific platform.

1.2.6 Details

Von Neumann's idea of a common memory for the program *and* the data seems obvious from today's point of view, but the earliest computers like Konrad Zuse's Z3 didn't work that way. In the Z3, for example, the memory for the program was a punch tape, decoupled from the input and output device, and from the main memory.

An interesting feature of the von Neumann architecture is that it allows self-modifying programs. These are popular among the designers of computer viruses, for example.

The von Neumann architecture with its two levels of memory (main memory and processor registers) is an idealized model, and we are implicitly working under this model throughout the course.

The reality looks more complicated. Modern computers also have a *cache*, logically belonging to the main memory, but allowing much faster access to memory cells (at the price of a more elaborate and expensive design). The idea is that frequently needed data are stored in the cache to speed up the program.

While caching is certainly a good thing, it makes the life of a programmer more difficult: you can no longer rely on the fact that access time to data is independent from where they are stored. In fact, to get the full performance benefit that caching can offer, the programmer has to make sure that data are accessed in a *cache-coherent* way. Doing this, however, requires some computer-specific knowledge about the cache, knowledge we were originally trying to avoid by using high-level programming languages. Luckily, we can often ignore this issue and (successfully) rely on the automatic cache management being offered. There is also a theoretical model for so-called *cache-oblivious* algorithms, in which an algorithm explicitly does not know the parameters of the cache. Algorithms which are efficient under this model, are (in a certain sense) efficient for any concrete cache size.

In real-life applications, we also observe the phenomenon that the data to be processed are too large to fit into the computer's main memory. Operating systems can automatically deal with this by logically extending the main memory to the hard disk. However, the *swapping* that takes place when hard disk data to be accessed are transfered to the main memory incurs a severe performance penalty, much worse than poor cache usage. In this situation, it is often useless to rely on the automatic mechanisms provided by the operating systems, and the programmer is challenged to come up with *input/output efficient* programs.

Even when we extend the von Neumann architecture to include several layers of memory, there are computers that don't fit in. Most notably, there are *parallel computers* with more than one processor. Writing efficient programs for such a computer is a task entirely different from programming for the von Neumann architecture. To take full advantage of the parallelism, programs have to be decomposed manually into independent parts, each of which is then run by one of the processors. In many cases, this is not at all a straightforward task, and specialized programming languages have to be used.

A recent successful alternative to parallel computers are networks of single-processor computers. You can even call this a computer architecture. Finally, there are *quantum computers* that are based on completely different physical principles than the von Neumann architecture. "Real" quantum computers cannot be built yet, but as a theoretical model, quantum computers exist, and algorithms are already being developed in this promising model of computation.

Chapter 2

Foundations

2.1 A first C++ program

The basic tool for the manipulation of reality is the manipulation of words. If you can control the meaning of words, you can control the people who must use the words.

Philip K. Dick, How to Build a Universe That Doesn't Fall Apart Two Days Later (1978)

This section presents a first complete C++ program and introduces the syntactical *and* semantical *terms necessary to understand all its parts.*

Here is our first C++ program. It asks for a number a as input and outputs its eighth power a^8. If you have never seen a C++ program before, even this short one might look scary, since it contains a lot of strange-looking symbols and words that are not found in natural language. On the other hand, this is good news: as short as it is, this program already contains many important features of the C++ language. Once we have gone through them in this section, this program (and even other, bigger programs) won't look scary anymore.

```
// Program: power8.C
// Raise a number to the eighth power.

#include <iostream>

int main()
{
  // input
  std::cout << "Compute a^8 for a =? ";
  int a;
  std::cin >> a;

  // computation
  int b = a * a; // b = a^2
  b = b * b;     // b = a^4

  // output b * b, i.e., a^8
  std::cout << a << "^8 = " << b * b << ".\n";
  return 0;
}
```

Program 1: *progs/power8.C*

If you compile this program on your computer and then run the executable file produced by the compiler, you find the following line on the standard output. Typically, the standard output is attached to some window on your computer screen.

```
Compute a^8 for a =?
```

You can now enter an integer, e.g. 2, using the keyboard. After pressing ENTER, the output on your screen reads as follows.

```
Compute a^8 for a =? 2
2^8 = 256.
```

Before discussing the program `power8.C` in detail, let us go over it once quickly. The lines starting with two slashes // are *comments*; they document the program such that it can easily be understood by a (human) reader. Line 4 contains an *include-directive*; in this case, it indicates that the program uses the input/output library `iostream`. The *main function* which is the heart of every C++ program spans lines 6–20. This function is called by the operating system when the program is started; it ends with a *return statement* in line 19. The value 0 is returned to the operating system, which by convention signals that the program terminated successfully.

The main function is divided into three parts. First, in lines 8–11 the input number is read. Line 9 outputs a message to the user that tells her which kind of input the program expects. In line 10 a *variable* `a` is declared that acts as a placeholder to store the input number. The keyword `int` indicates that `a` is an integer. In line 11, finally, the variable `a` receives its value from the input.

Then in lines 13–15 the actual computation takes place. In line 14, a new variable `b` is declared which acts as a placeholder to store the result of the computation. The variable `b` is initialized to the product `a * a`. Line 15 computes the product `b * b`, that is, a^4 and stores this result again in `b`.

The third part in lines 17–18 provides the program output. Part of it is the computation of the product `b * b`, that is, a^8.

2.1.1 Syntax and semantics.

In order to understand the program `power8.C` in detail, and more importantly, to write programs yourself later, you need to know the rules according to which programs are written. These rules form the *syntax* of C++. You further need to know how to interpret a program ("what does the program do?"), and this is determined by the *semantics* of C++. Even a program that is well-formed according to the C++ syntax may be *invalid* from a semantical point of view. A *valid* program is one that is syntactically *and* semantically correct.

It's the same with natural language: grammar tells you what sentences are, but the interpretation of a sentence (in particular, whether it makes sense at all) requires a concept of meaning.

When a program is invalid, the compiler may output an error message, and this will definitely happen when the program contains *syntax errors*, violations of the syntactical rules. A program that is semantically invalid may compile without errors, but we are not allowed to make any assumptions about its behavior; the program *could* run fine, for example if the semantical error in question has no consequences on a particular platform. On other platforms, the program may behave strangely, or crash. Even on the same platform, it might work sometimes, but fail at other times. We say that the program's behavior is *undefined*. Clearly, one should avoid writing programs that exhibit undefined behavior.

The syntax of C++ is specified formally in a mathematical language. The description of the semantics is less strict; it rather resembles the text of a law, and as such it suffers from omissions and possible misinterpretations. The official law of C++ covering both syntax and semantics, is the ISO/IEC standard 14882 from 1998.

While such a formal specification is indispensable (otherwise, how should a compiler know whether your program text is actually a C++ program, and what it is supposed to do?), it is not suitable for learning C++. Throughout this book, we explain the relevant syntactical and semantical terms in natural language and by example. For the sake of readability, we will often not strictly distinguish between syntactical and semantical terms: some terms are most naturally introduced as having both syntactical and semantical aspects, and it depends on the context which aspect is relevant.

Unspecified and implementation defined behavior. Sometimes, even valid programs behave differently on different platforms; this is one of the more ugly aspects of C++ that we'd prefer to sweep under the rug. Unfortunately, we can't ignore the issue completely, since it occasionally pops up in "real life".

There are two kinds of platform-dependent behavior. The nicer one is called *implementation defined* behavior.

Whenever the C++ standard calls some aspect of the language "implementation defined", you can expect your platform to contain documentation that fully specifies the aspect. The typical example for such an an implementation defined aspect is the number of bits that make up a memory cell, see Section 1.2.3. In case of implementation defined aspects and resulting behavior, the C++ standard and the platform together completely determine the actual behavior.

The less nice kind is called *unspecified behavior*, coming from some unspecified aspect of the language. Here you can rely on a well-defined and usually *small* set of possible specifications, but the platform is not required to contain a full specification of the aspect. A typical example for such an unspecified aspect is the evaluation order of operands within an expression, see Section 2.1.11.

In writing programs, unspecified aspects cannot always be avoided, but usually, some care ensures that no unspecified or even undefined behavior results.

2.1.2 Comments and layout

Every good program contains comments, for example

```
// Program: power8.C
// Raise a number to the eighth power.
```

A comment starts with two slashes // and continues until the end of the line. Comments do not provide any functionality, meaning that the program would do exactly the same without them. Why is a program without comments bad, then? We do *not* only write programs for the compiler to translate them into executables, but we also write them for other people (including ourselves) to read, modify, correct or extend them.

Without comments, the latter tasks become very tedious when the program is not completely trivial. Trust us: Even you will not be able to understand your own programs after a couple of weeks, without comments. There is no *standard* way of writing comments, but we will follow some common-sense guidelines. One of them is that every program—even if it is very simple—should start with one or more lines of comments that mention the program's name and say what it does. In our case, the above two lines fully suffice.

Another key feature of a readable program is its layout; consider the version of `power8.C` shown in Program 2. We have removed comments, and all "unnecessary" layout elements like spaces, line breaks, blank lines, and indentations.

```
#include <iostream>
int main(){std::cout<<"Compute a^8 for a =? ";
int a;std::cin>>a;int b=a*a;b=b*b;std::cout<<
a<<"^8 = "<<b*b<<".\n";return 0;}
```

Program 2: *progs/power8_condensed.C*

The compiler is completely ignorant about these changes, but a person reading the program will find this condensed version quite difficult to understand. The purpose of a good layout is to visualize the program structure. This for example means that logical blocks of the program should be separated by blank lines, or that one line of sourcecode should be responsible for only one thing. *Indentation*, like `power8.C` has it between the pair of curly braces, is another indispensable ingredient of good layout, although you will only later be able to fully appreciate this.

Typically, collaborative software projects have layout guidelines, making sure that everybody in the project can easily read everybody else's code. At the level of the simple programs discussed in this book, such formal guidelines are not necessary; we simply adhere to standard guidelines that have proven to work well in practice, and that are being used in almost any other book on C++ as well.

2.1.3 Include directives

Every useful program contains one or more `include` *directives*, such as

```
#include <iostream>
```

Usually, these appear at the very beginning of the program. `#include` directives are needed since, in C++, many important features are not part of the core language. Instead, they are implemented in the so-called *standard library* which is part of every C++ implementation. A *library* is a logical unit used to group certain functionality and to provide it to the user in a succinct form. In fact, the standard library consists of several libraries one of which is the input/output library.

A library presents its functionality to the user in the form of one or several *headers*. Each such header contains information that is needed by the compiler. In order to use a certain feature from a library, one has to include the corresponding header into the program by means of an `#include` directive. In `power8.C`, we want to use input and output which are (maybe surprisingly) not part of the core language. The corresponding header of the standard library is called `iostream`.

A well-designed C++ library puts its functionality into a *namespace*. The namespace of the standard library is called `std`. Then, in order to access a feature from the library, we have to *qualify* its name with the namespace, like in `std::cin` (this is the feature that allows us to read input from the keyboard). This mechanism helps to avoid *name clashes* in which different features accidentally get the same name. At the same time, explicit qualification increases the readability of a program, as it is immediately apparent from which library a given feature comes. A name that is not qualified is called *unqualified* and usually corresponds to a feature defined in our own program.

2.1.4 The main function

Every C++ program must have a `main` function. The shortest program reads as follows.

```
int main() { return 0; }
```

This program does nothing. The `main` function is called by the operating system when you tell it to run the program; but why is it a *function*, and what is "`return 0;`" supposed to mean? Just like a mathematical function, the `main` function can have arguments given to it upon execution of the program, and the computations within the curly braces yields a function value that is given back (or returned) to the operating system. In our case, we have written a `main` function that does not expect any arguments (this is indicated by the empty brackets `()` behind `main`) and whose return value is the integer `0`. The fact that the return value must be an integer is indicated by the word `int` before `main`. By convention, `return 0` tells the operating system that the program has run successfully (or that we don't care whether it has), while any other value explicitly signals failure.

In a strict mathematical sense, the `main` function of `power8.C` is utterly boring. The whole functionality of the program comes from the *effect* of the function. This effect is to read a number from the standard input and write its eighth power to the standard output. The fact that functions can have effects sets C++ apart from many *functional programming languages*.

2.1.5 Values and effects

The value and effect of a function are determined by the C++ semantics. Merely knowing the syntactical rules of writing functions does not tell us anything about values and effects. In this sense, value and effect are purely semantical terms.

For example, we have to *know* that in C++, the character `0` is interpreted as the integer 0 (although this is not difficult to guess). It is also important to understand that value and effect depend on the concrete program state in which the function is called.

2.1.6 Types and functionality

The word `int` is the name of a C++ type. This type is used since the program `power8.C` deals with integers. In mathematics, integers are modeled by the ring $(\mathbb{Z}, +, \cdot)$. This algebraic structure defines the integers in terms of their value range (the set $\mathbb{Z}$), and in terms of their functionality (addition and multiplication). In C++, integers can be modeled by the type `int`. Like a "mathematical type", a C++ type has a *name*, a *value range*, and *functionality*, defining what we can do with it. When we refer to a type, we will do so by its name. Note that the name is a syntactical aspect of the type, while value range and functionality are of semantical nature.

Conveniently, C++ contains a number of *fundamental types* (sometimes called built-in types) for typical applications. The type `int` is one of them. The major difference to the "mathematical type" $(\mathbb{Z}, +, \cdot)$ is that `int` has a finite value range only.

2.1.7 Literals

A literal represents a constant value of some type. For example, in line 19 of the program `power8.C`, `0` is a literal of type `int`, representing the value 0. For each fundamental type, it is separately defined how its literals look like, and what their values are. A literal can be seen as the syntactical counterpart of a value: it makes the value "visible" in the program.

2.1.8 Variables

The line

```
int a;
```

is a *declaration* of a *variable*. A variable represents a not necessarily constant value of some type. The variable has a *name*, a *type*, a *value*, and an *address* (typically in the computer's main memory; you can think of the address simply as the position of the variable in the main memory). The purpose of the address is to know where to store and look up the value. The reason for calling such an entity a *variable* is that its value can be changed by modifying the memory content at the corresponding address. The address itself may change as well. In contrast, the name and type remain fixed.

When we refer to a variable, we will do so by its name. The declaration `int a;` *defines* a variable with the following characteristics.

name	type	value	address
a	int	undefined	chosen by compiler/OS

You might wonder why this is called a *definition* of a, even though it does not define the value of a. But recall that this value depends on the program state, and that the definition fully specifies *how* the value is obtained: look it up at a's address. Saying that a variable *has* a value is therefore somewhat imprecise, but we'll stick to it, just like mathematicians talk about *function value* when they actually mean the value obtained by evaluating the function with concrete arguments. We even go one step further with our sloppiness: if a has value 2, for example, we also say that "a is 2". This is the way that programmers usually talk about variables and their values. We will get to know mechanisms for assigning and changing values of variables in Section 2.1.13.

2.1.9 Identifiers and names

The name of any variable must be an *identifier*, according to the following definition, and it must be different from certain *reserved* names like `int`.

Definition 1 *An* **identifier** *is any sequence of characters composed of the 52 letters* `a...z` *and* `A...Z`*, the 10 digits* `0...9`*, and the underscore (_). The first character has to be a letter.*

A C++ program may also contain other names, for example the *qualified* names `std::cin` and `std::cout`. The C++ syntax specifies what a name is, while the C++ semantics tells us what the respective name refers to in a given context.

2.1.10 Objects

An object is a part of the computer's main memory that is used by the program to store a value. An object has an address, a type, and a value of its type (determined by the memory content at the object's address).

With this definition, a variable can be considered as a named object, but we may also have unnamed objects. Although we can't show an example for an unnamed object at this point, we can argue that unnamed objects are important.

In fact, if you want to write interesting programs, it is absolutely necessary to work with objects that are not named by variables. This can be seen by the following simple thought experiment: suppose that you have written a program that stores a sequence of integers to be read from a file (for example, to sort them afterwards). Now you look at your program and count the number of variables that it contains. Say this number is 31. But in these 31 variables, you can store no more than 31 integers. If your program is of any practical use, it can certainly store a sequence of 32 integers, but then there must be at least one integer that cannot be stored under a variable name.

2.1.11 Expressions

In the program `power8.C`, three character sequences stand out, because they look familiar and are chiefly responsible for the functionality of the program: these are the character sequences `a * a` in line 14 and `b * b` in lines 15 and 18.

An expression represents a computation involving other expressions. More precisely, an expression is either a *primary expression*, for example a literal or a name, or it is a *composite expression*. A composite expression is obtained by combining expressions through certain operations, or by putting a pair of parentheses `()` around an expression.

The expression `a * a` is an *arithmetic* expression, involving *numeric* variables[1] and the multiplication operator, just like we know it from mathematics. According to our above definition, `a * a` is a composite expression, built from the multiplication operator and the two primary expressions `a` and `a`.

According to the above definition, an expression is a syntactical entity, but it has semantical aspects as well: any expression has a type, a value of this type, and possibly an effect. The type is fixed, but the value and the effect only materialize when the expression gets *evaluated*, meaning that the computation it represents is carried out. Evaluating an expression is the most frequent activity going on while a C++ program is executed; the evaluation computes the value of the expression and carries out its effect (if any).

Type and value of a primary expression are determined by its defining literal, or by type and value of the entity behind its defining name. Primary expressions have no effect. Type, value and effect of a composite expression are determined by the involved operation, depending on the values and effects of the involved sub-expressions. Putting parentheses `()` around an expression yields an expression with the same type, value and effect.

The expression `a * a`, for example, is of type `int`, and not unexpectedly, its value is the square of the value of `a`. The expression has no effect. The expression `b = b * b`, built from the *assignment operator* and the two expressions `b` and `b * b`, has the same type and value as `b * b`, but it has an additional effect: it assigns the square of `b` back to `b`.[2]

We say that an expression is *evaluated* rather than *executed*, because many expressions do not have an effect, so that their functionality is associated with the value only. Even for expressions with effect, some books use the term *side effect* to emphasize that the important thing is the value. The C++ entities chiefly responsible for effects are the *statements* to which we get below.

We want to remark that the *only* way of accessing an expression's value is to evaluate it, and this also carries out its effect. You cannot get the value without the effect.

[1] The truth is that the expression involves the *names* of the variables, but for the sake of readability, we suppress this subtlety.

[2] This is a shorthand for the correct, but somewhat clumsy formulation that the new value of `b` is set to the square of the old value of `b`.

2.1.12 Lvalues and rvalues

An lvalue is an expression that has an address. In the program `power8.C`, the variable `b` is an lvalue, and its address is the address of the variable `b`.

The value of an lvalue is defined as the value of the object at its address. An lvalue can therefore be viewed as the syntactical counterpart of an object: it gives the object a (temporary) name and makes it "visible" within a C++ program. We also say that the lvalue *refers* to the object at its address.

In particular, any variable is an lvalue. But lvalues provide a means for accessing and changing object values, even without having a corresponding variable. As we will see in Section 2.1.13 below, the expression `std::cout << a` "hidden" in line 18 is such an lvalue.

Any expression that is not an lvalue is an rvalue. For example, literals are rvalues: there is no address associated with the `int`-literal `0`, say. Putting a pair of parenthesis around an lvalue yields an lvalue, and similarly for rvalues.

The terms l*value* and r*value* already indicate that we think about them not so much in terms of expressions, but rather in terms of their values. We will often identify an lvalue with the object it refers to, and an rvalue simply with its value.

2.1.13 Operators

Line 14 of `power8.C`, for example, features the binary multiplication operator `*`.

Like a function, an operator expects arguments (here also called *operands*) of specified types, from which it computes a *return value* of a specified type, according to its *functionality*. In addition, these computations may have an effect.

This was the semantical view; on the syntactical level, the operands as well as the composite expression (built from the operator and its operands, see Section 2.1.11), are expressions; the operator specifies for each of them whether it is an lvalue or an rvalue. If the composite expression is an lvalue, the operator is said to return the object referred to by the lvalue. If the composite expression is an rvalue, the operator simply returns its value.

The number of operands is called the *arity* of the operator. Most operators have arity 1 (unary operators) or 2 (binary operators).

Whenever an rvalue is expected as an operand, it is also possible to provide an lvalue. In this case, the lvalue will simply be interpreted as an rvalue, meaning that its address is only used to look up the value, but *not* to change it. This is known as *lvalue-to-rvalue conversion*. In stating that an operand must be an rvalue, the operator therefore guarantees that the operand's value remains unchanged; by expecting an lvalue, the operator explicitly signals its intention to change the value.

Evaluation of composite expressions. When a composite expression involving an operator gets evaluated, the operands are evaluated first (recall that this also carries out the effects of the operands, if any). Based on the resulting values, the operator computes the value of the composite expression. The latter computations may have additional effects, and all effects together form the effect of the composite expression.

The order in which the operands of a composite expression are evaluated is (with rare exceptions) unspecified, see also Section 2.1.1.

Therefore, if the effect of one operand influences values or effects of other operands, value and effect of the composite expression may depend on the evaluation order. The consequence is that value and effect of the composite expression may be unspecified as well.

Since the compiler is not required to issue a warning in such cases, it is the responsibility of the programmer to avoid any expression whose value or effect depends on the evaluation order of operands.

Operator specifics. What is it that sets operators apart from functions? On the one hand, there is only a finite number of possible operator *tokens* such as + or =. Many of these tokens directly correspond to well-known mathematical operator symbols indicating the functionality of the operator.[3] On the other hand, and most conveniently, operator calls do not have to obey the usual function call notation, like in `f(x,y)`. After all, we want to write `a * a` in a program, and not `*(a,a)`. In summary, operators let us write more natural and more readable code.

Four different operators (all of them binary) occur in `power8.C`, namely the multiplication operator `*`, the assignment operator `=`, the input operator `>>`, and the output operator `<<`. Let us discuss them in turn.

Multiplication operator. The multiplication operator `*` expects two rvalue operands of some type T, and it returns the product of its two operands as an rvalue. The multiplication operator has no effect on its own.

Assignment operator. The assignment operator `=` expects an lvalue of some type T as its first operand, and an rvalue of the same type as its second operand. It assigns the value of the second operand to the first operand and returns the first operand as an lvalue. In our program `power8.C`, the expression `b = b * b` therefore sets the value of `b` to the square of its previous value, and then returns `b`.

In fact, the letter "l" in the term lvalue stands for the fact that the expression may appear on the *left* hand side of an assignment operator. Similarly, the term rvalue signals an expression that may appear only on the *right* hand side of an assignment operator.

Input Operator. In `power8.C`, the composite expression `std::cin >> a` in line 11 sets the variable `a` to the next value from the *standard input*, usually the keyboard.

[3]Unfortunately, the token = corresponds to mathematical assignment $:=$, and not to mathematical equality $=$, a constant source of confusion for beginners.

In general, the input operator >> expects as its first operand an lvalue referring to an *input stream*. The second operand is an lvalue of some type T. The operator sets the second operand to the next value read from the input stream and returns the stream as an lvalue.

An input stream represents the state of some input device. We think of this device as producing a continuous stream of data that can be tapped to provide input on demand. Under this point of view, the state of the stream corresponds to the sequence of data not yet read. In setting the value of its second operand, the input operator removes one data item from the stream to reflect the fact that this item has now been read. For this, it is important that the stream comes as an lvalue. Conceptually, an input stream is also considered part of the program state.

How much of the data is read as one item, and how exactly it is interpreted as a value of type T highly depends on the type T of the second operand. For now, it is enough to know that this interpretation is readily defined for the type `int` and for the other fundamental types that we will encounter in the following sections.

In C++, the lvalue `std::cin` refers to the variable `cin` defined in the input/output library, and this variable corresponds to the standard input stream.

It is up to the program's caller to fill the standard input stream with data. For example, suppose that the program was started from a command shell. Then usually, while the program is running, all input to the command shell is forwarded to the program's standard input stream. It is also possible to redirect a program's standard input stream to read data from a file instead.

The fact that the input operator returns the input stream is not accidental, as it allows to build expressions involving chains of input operations, such as `std::cin >> x >> y`. We will discuss this mechanism in detail for the output operator below.

Output Operator. In `power8.C`, the composite expression `std::cout << a` in line 18 writes the value of a to the *standard output*, usually the computer screen.

In general, the output operator << expects as its first operand an lvalue referring to an *output stream*. The second operand is an rvalue of some type T. The operator writes the value of the second operand to the output stream and returns the output stream as an lvalue.

An output stream represents the state of some output device. We think of this device as storing the continuous stream of output data that is generated by the program. In writing to the stream, the output operator therefore changes the stream state, and this makes it necessary to provide the stream as an lvalue. Conceptually, an output stream is also considered part of the program state.

It depends on the type T in which format the second operand's value is written to the stream; for the type `int` and the other fundamental types, this format is readily defined.

C++ defines a standard output stream `std::cout` and a *standard error* stream `std::cerr` in the input/output library.

It is up to the program's caller to process these output streams. For example, suppose that the program was started from a command shell. Then usually, while the program is running, both standard output stream and standard error stream are forwarded to the command shell. But it is also possible to redirect one or both of these streams to write to a file instead. This can be useful to separate regular output (sent to `std::cout`) from error output (sent to `std::cerr`).

As indicated above for input streams, it is possible to output several values through one expression, as in

```
std::cout << a << "^8 = " << b * b << ".\n"
```

Maybe this looks a bit strange, because there is more than one << operator token and more than two operands; but in mathematics, we also write $a + b + c$ as a shortcut for either $(a + b) + c$ or $a + (b + c)$; because addition is associative, we don't even have to specify which variant we intend.

In C++, such shortcuts are also allowed in order to avoid cluttering up the code with parentheses. But C++ operators are in general not associative, so we have to know the 'logical parentheses' in order to understand the meaning of the shortcut.

The operators >> and << are *left-associative*, meaning that the above expression is logically parenthesized as follows.

```
(((std::cout << a) << "^8 = ") << b * b) << ".\n"
```

Recall that the innermost expression `std::cout << a` is an lvalue referring to the standard output stream. Hence, this expression serves as a legal first operand for the next outer composite expression `(std::cout << a) << "^8 = "` and so on. The full expression therefore outputs the values of *all* expressions occurring after some <<, from left to right. The rightmost of these expressions ends with \n which encodes a line break (newline).

2.1.14 Statements

A statement is a basic building block of a C++ program, and it possibly has an effect. The effect depends on the program state and materializes when the statement is *executed.* As with expressions, we say that a statement *does* something. A statement usually ends with a semicolon and represents a "step" of the program. Statements are executed in top-to-bottom order. The shortest possible statement is the *null statement* consisting only of the semicolon; it has no effect. In a typical program, most statements evaluate one or several expressions.

A statement is not restricted to one line of sourcecode; on the contrary, readability often requires to break up statements into several lines of code. The compiler ignores these line breaks, as long as we do not put them at unreasonable places like in the middle of a name.

In `power8.C`, there are three kinds of statements.

Expression statement. Appending a semicolon to an expression leads to an expression statement. It evaluates the expression but does not make use of its value. This is a frequent form of statements, and in our small program, the statement

```
b = b * b;
```

as well as all statements starting with `std::cin` or `std::cout` are expression statements.

Declaration statement. Such a statement introduces a new *name* into the program. This can be the name of a variable of a given type, like in the declaration statements

```
int a;
```

and

```
int b = a * a;
```

A declaration statement consist of a *declaration* and a concluding semicolon. In our program `power8.C`, we deal with *variable* declarations; they can be of the form

$T\ x$

or

$T\ x = expr$

where T is a type, x is the name of the new variable, and *expr* is an rvalue of type T. A variable declaration is different from an expression; for example, it can occur at specific places only. But when it occurs, it behaves like an expression in the sense that a declaration also has an effect and a value. Its effect is to allocate memory for the new variable at some address, and to *initialize* it with the value of *expr*, if present. Its value is the resulting value of the new variable. The declaration is said to *define* the variable.

As in the case of expression statements, a declaration statement carries out the effect of the declaration and ignores its value.

Return statement. Such a statement is of the form

`return` *expression*;

where *expression* is an rvalue. It only occurs within a function. The return statement evaluates *expression*, finishes the function's computations, and puts *expression*'s value at some (temporary) address that the caller of the function can access. Abstracting from these technicalities, we simply say that the statement *returns expression* to the caller.

We have seen only one example so far: the statement `return 0;` returns the value 0 (formally, the literal `0` of value 0) to the operating system which has called the `main` function of our program.

Figure 2 summarizes the syntactical and semantical terms we have introduced, along with their relations. The figure emphasizes the central role that expressions play in C++.

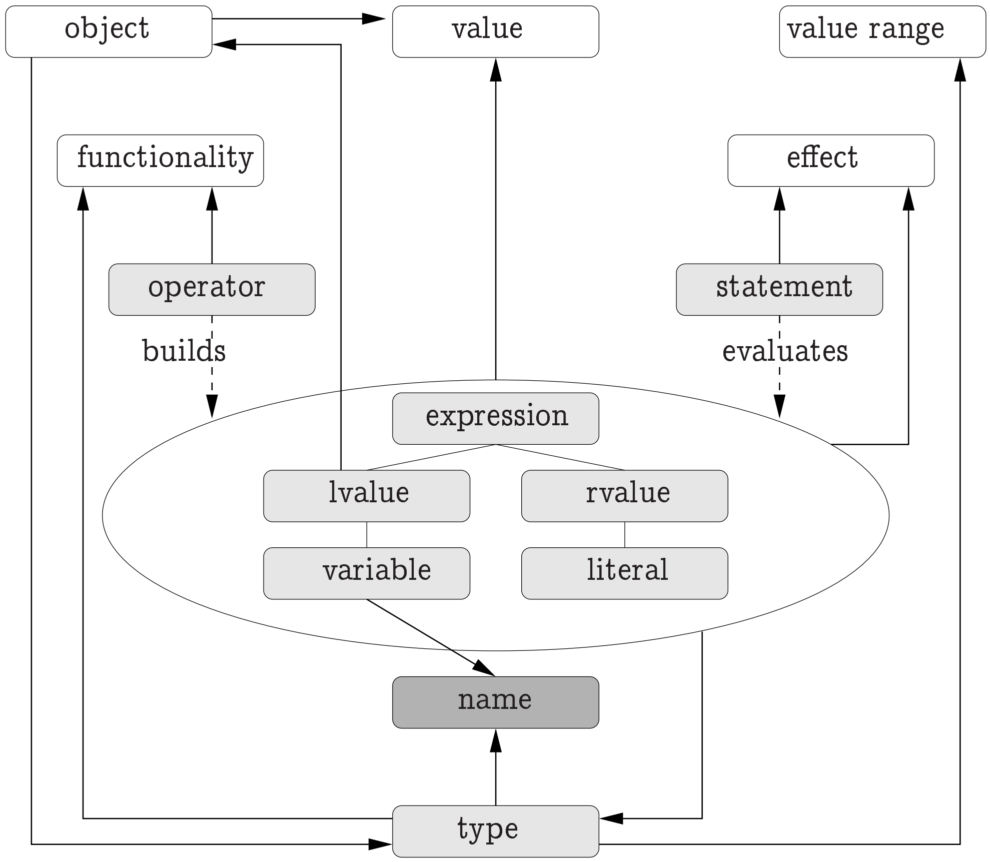


Figure 2: *Syntactical and semantical terms appearing in our first program* `power8.C`. *Purely semantical terms appear in white, purely syntactical terms in dark gray. Mixed terms are drawn in light gray. Solid arrows* A $\longrightarrow$ B *are to be read as "A has B", while solid lines in the expression part mean that the upper term is more general than the lower one.*

2.1.15 The first program revisited

If you run the executable file resulting from Program 1 for a couple of input values, you will quickly notice that something is weird. For example, on the platform of the authors, the following happens:

```
Compute a^8 for a =? 15
15^8 = -1732076671.
```

Obviously, the eighth power of a positive number cannot be negative, so what is going on? We will discuss this in detail in the next section, but the short explanation is that the type `int` can only deal with numbers up to a certain size. If the mathematical result of a computation exceeds this size, the C++ result will necessarily differ from the mathematical result.

This sounds like bad news; after all, 15^8 is not *such* a big number, and if we cannot even compute with numbers of this size, what can we compute at all? The good news is that the problem is easy to fix. The authors have implemented a type called `ifm::integer` that is capable of dealing with integers of *arbitrary* size (up to the memory limits, of course). Using this type is very easy, and this is one of the key strengths of C++: we simply have to replace `int` by `ifm::integer` in our program and in addition include the definition of the new type. Here is the accordingly changed program.

```
// Program: power8_exact.C
// Raise a number to the eighth power,
// using integers of arbitrary size

#include <iostream>
#include "integer.h"

int main()
{
  // input
  std::cout << "Compute a^8 for a =? ";
  ifm::integer a;
  std::cin >> a;

  // computation
  ifm::integer b = a * a; // b = a^2
  b = b * b;              // b = a^4

  // output b * b, i.e., a^8
  std::cout << a << "^8 = " << b * b << ".\n";
  return 0;
}
```

Program 3: *progs/power8_exact.C*

In order for this to compile, you need to have the file `integer.h` in your working directory (the one that also contains `power8_exact.C`).

Using the above program, you *can* compute the correct value of 15^8:

```
Compute a^8 for a =? 15
15^8 = 2562890625.
```

But also much larger values will work (if you happen to be interested in them):

```
Compute a^8 for a =? 1234567
1234567^8 = 5396563761318393964062660689603780554533710504641.
```

We will not discuss the type ifm::integer any further in this book, and there's no need for it, since it just works like int (except that it does not have the size limitations of int). But whenever you need (in an exercise or challenge) larger numbers, you are free to use the type ifm::integer.

2.1.16 Details

Commenting. There is a way of writing comments that are not limited to one line of code. Any text enclosed by /* (start of comment) and */ (end of comment) is ignored by the compiler. The initial comment of our program power8.C could also have been written as

```
/*
  Program: power8.C
  Raise a number to the power 8.
*/
```

This mechanism may seem useful for longer comments spanning several lines of code, but the problem is that you do not immediately recognize a line in the middle of such a construction as a comment: you always have to look for the enclosing /* and */ to be sure.

Sometimes, /* and */ are used for very short comments within lines of code, like in

```
c = a + /* don't subtract! */ b;
```

For readability reasons, we do not advocate this kind of comment, either.

Identifiers starting with an underscore. Occasionally, real-life C++ code contains "identifiers" starting with the underscore character _, although this is not allowed according to Definition 1. The truth is that *the programmer* is not allowed to use such "identifiers"; they are reserved for internal use by the compiler. Compilers should issue at least a warning, when they discover such a badly formed "identifier", but often they just let it pass.

Define variables where needed! In C++, it is good general practice to define a variable immediately before it is used the first time. The readability of a program improves, if for any variable that appears in some expression, the corresponding definition is nearby and can hence be found quickly. This guideline is in contrast to some other programming languages; for example, in C all variables have to be declared at the beginning of the function where they are used.

The main function. The `main` function is an exceptional function in several ways. One particular specialty is that the return statement can be omitted. A `main` function without a return statement at the end behaves precisely as if it would end with `return 0;`. This definition has been made for historical reasons mostly; it is an anomaly compared to other functions (which will be discussed later). Therefore, we stick to the explicit return statement and ask you to do the same.

Using directives. It is possible to avoid all `std::` prefixes through one additional line of code, a `using` *directive*. In case of `power8.C`, this would look like in Program 4.

```
// Program: power8.C
// Raise a number to the eighth power.

#include <iostream>

using namespace std;

int main()
{
  // input
  cout << "Compute x^8 for x =? ";
  int a;
  cin >> a;

  // computation
  int b = a * a; // b = a^2
  b = b * b;     // b = a^4

  // output b * b, i.e., a^8
  cout << a << "^8 = " << b * b << ".\n";
  return 0;
}
```

Program 4: *progs/power8_using.C*

The `using` directive is a declaration statement of the form

`using namespace` *X*;

It allows us to use all features from namespace X without qualifying them through the prefix `X::`. This mechanism seems quite helpful at first sight, but it has severe drawbacks that prevent us from using (let alone advocating) it in this book.

Let's start with the major drawback. Namespaces may have a large number of features (in particular, the namespace `std` has), with a large number of names. `cin` and `cout` are two such names from the namespace `std`. It is very difficult (and also not desirable) to know all these names. On the other hand, it *would* be good to know them in order to avoid conflicts with the names *we* introduce. For example, if we define a variable named `cout` somewhere in Program 4, we are asking for trouble: when we later use the expression `cout`, it is not clear whether it refers to the standard output stream, or to our variable. We can easily avoid the variable name `cout`, of course, but we may accidentally introduce another name that also appears in the namespace `std`. The unfortunate consequence is that in some expression of our program, this name might *not* refer to the feature we introduced, but to a feature of the same name from the standard library. We may end up silently using a feature from the standard library that we don't even know and that we never intended to use. The resulting strange behavior of our program can be very difficult to track down.

In the original program `power8.C`, introducing a name `cout` (or any other name also appearing in namespace `std`) does not cause any harm: without the `std::` qualification, it can never "accidentally" refer to something from the standard library.

Here is the second drawback of `using` directives. A large program contains many names, and in order to keep track of, it is desirable that the name "tells" us where it comes from: is it a name we have introduced, or does it come from a library? If so, from which one? With `using` directives, we lose that information, meaning that the program becomes less readable and more difficult to maintain.

2.1.17 Goals

Dispositional. At this point, you should ...

1) understand the basic syntactical and semantical terms of C++, in particular *expression*, *operator*, *statement*, *lvalue*, *rvalue*, *literal*, and *variable*;
2) understand syntax and semantics of the program `power8.C`.

Operational. In particular, you should be able to ...

(G1) describe the basic syntactical and semantical terms of C++ (as listed above) in your own words and give examples;

(G2) tell whether a given character sequence is an identifier;

(G3) tell whether a given character sequence is a simple expression, as defined below;

(G4) find out whether a given simple expression is an lvalue or an rvalue;

(G5) evaluate a given simple expression;

(G6) read and write programs with functionality similar to `power8.C`.

A *simple expression* is an expression which only involves `int`-literals, identifiers, the binary multiplication operator, the assignment operator, and parentheses.

2.1.18 Exercises

Exercise 1 *Which of the following character sequences are not C++ identifiers, and why not?* (G2)

(a)	`identifier`	*(b)*	`int`	*(c)*	`x_i`	*(d)*	`4x__`
(e)	`A99_`	*(f)*	`_tmp`	*(g)*	`T#`	*(h)*	`x12b`

Exercise 2 *Which of the following character sequences are not C++ expressions, and why not? Here,* `a` *and* `b` *are variables of type* `int`*.* (G3)

(a)	`1*(2*3)`	*(b)*	`a=(b=5)`	*(c)*	`1=a`	*(d)*	`(a=1)`
(e)	`(a=5)*(b=7)`	*(f)*	`(1`	*(g)*	`(a=b)*(b=5)`	*(h)*	`(a*3)=(b+5)`

Exercise 3 *For all of the expressions that you have identified in Exercise 2, decide whether these are lvalues or rvalues, and explain your decisions.* (G4)

Exercise 4 *Determine the values of the expressions that you have identified in Exercise 2 and explain how these values are obtained. Which of these values are unspecified and can therefore not be determined uniquely?* (G5)

Exercise 5 *What is the smallest natural number that is divisible by all numbers between* 2 *and* (G6)

a) 10 ?

b) 20 ?

c) 30 ?

The result is also known as the *least common multiple* of the respective numbers.

Note: This exercise does not require you to write a program, but you may use a program to help you in the computations.

Exercise 6 *Write a program* `multhree.C` *that reads three integers* a, b, c *from standard input and outputs their product* abc*.* (G6)

Exercise 7 *Write a program* `power20.C` *that reads an integer* a *from standard input and outputs* a^{20} *using at most five multiplications.* (G6)

Exercise 8 *During an electronic transmission, the following C++ program got garbled. As you can see, the layout got messed up, but at the same time, some errors got introduced as well.* (G4)(G6)

```
#include <iostream> int main[]{int a;int b;int c;std::cin >> a;
cin >> b;c = a * b;std::cout << c*c;return 0;}
```

a) Write the program down in a well-formatted way.

b) The program contains two syntax errors. Fix them! What does the fixed program do?

c) The (fixed) program contains a number of composite expressions. List them all, and decide for each composite expression whether it is an rvalue or an lvalue. Recall that a composite expression may consist of primary expressions, but also of other composite expressions.

d) Enhance the program with informative output so that it becomes easier to use.

e) Add sensible comments to the program; most notably, there should be a comment in the beginning that says what the program does.

2.1.19 Challenges

Exercise 9 *The "slow" method for computing the eighth power of an integer a needs seven multiplications. Program 1 requires only three, and we believe that this should be faster. The goal of this challenge is to find out: how much faster? For example, if we compute the eighth power of a $10,000$-digit number using both methods, what will be the difference in runtime? Using the type* `int`*, though, we cannot correctly compute with $10,000$-digit numbers (as you will easily notice when you start* `power8.C` *with somewhat larger inputs, see Section 2.1.15). For this reason, you should use the type* `ifm::integer` *for your computations.*

Write two programs, `power8_slow.C` *and* `power8_fast.C` *that compute the eighth power of an integer with 7 and 3 multiplications, respectively (over the exact type* `ifm::integer`*). Since we want to measure runtimes, there should be no output (you don't want to read it, anyway). In order to be able to use the same large inputs for both programs, it is beneficial to have the programs read the input from a file. For example, if you have a file* `power8.dat` *with contents* `1234567`*, you can tell the program* `power8_exact.C` *to read its input from this file by starting it with the command*

```
./power8_exact < power8.dat
```

Now that you have both programs, create an input file `power8.dat`*, and fill it with larger and larger numbers (each time doubling the number of digits, for example). Then measure the times taken by each of the programs* `power8_slow.C` *and* `power8_fast.C` *on these inputs. You can simply do this using your watch (for sufficiently many digits both programs will be slow enough), or you can start the programs like this:*

```
time ./power8_fast < power8.dat
```

This command will run the program and afterwards output the number of seconds that it took (the first number, the one ending in `u`*, is the relevant one).*

What do you observe? Is power8_fast.C *around twice as fast as* power8_slow.C *(this is what you might expect from the number of multiplications)? Or do you observe a speedup factor different from 2? And is this factor stable as the input numbers get larger?*

Whatever you observe, try to explain your observations!

Exercise 10 *Let $\ell(n)$ be the smallest number of multiplications that are needed in order to compute the n-th power a^n of an integer a. Since $\ell(n)$ may depend on what we consider as a "computation", we make $\ell(n)$ well-defined by restricting to the following kind of computations. Let a_0 denote the input number a. A computation consists of t steps, where t is some natural number, and step $i, 1 \leq i \leq t$ has the form*

```
a_i = a_j * a_k
```

with $j, k < i$. The computation is correct if $a_t = a^n$. For example, to compute a^8 in three steps (three multiplications) as in power8.C, *we can use the computation*

```
a_1 = a_0 * a_0
a_2 = a_1 * a_1
a_3 = a_2 * a_2
```

Now, $\ell(n)$ is defined as the smallest value of t such that there exists a correct t-step computation for a^n.

a) *In the above model of computation, prove that for all $n \geq 1$,*

$$\lambda(n) \leq \ell(n) \leq \lambda(n) + \nu(n) - 1,$$

where $\lambda(n)$ is one less than the number of significant bits of n in the binary representation of n (see Section 2.2.8), and $\nu(n)$ is the number of 1's in the binary representation of n. For example, the binary representation of 20 is 10100, and hence $\lambda(20) = 4$ and $\nu(20) = 2$, resulting in $\ell(n) \leq 5$.

b) *Either prove that the upper bound in a) is always best possible, or find a value n^* such that $\ell(n^*) < \lambda(n^*) + \nu(n^*) - 1$.*

2.2 Integers

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.

Leopold Kronecker, in a lecture to the Berliner Naturforscher-Versammlung (1886)

This section discusses the types int *and* unsigned int *for representing integers and natural numbers, respectively. You will learn how to evaluate arithmetic expressions over both types. You will also understand the limitations of these types, and—related to this—how their values can be represented in the computer's memory.*

Here is our next C++ program. It asks the user to input a temperature in degrees Celsius, and outputs it in degrees Fahrenheit. The conversion is defined by the following formula.

$$\text{Degrees Fahrenheit} = \frac{9 \cdot \text{Degrees Celsius}}{5} + 32.$$

```
// Program: fahrenheit.C
// Convert temperatures from Celsius to Fahrenheit.

#include <iostream>

int main()
{
  // Input
  std::cout << "Temperature in degrees Celsius =? ";
  int celsius;
  std::cin >> celsius;

  // Computation and output
  std::cout << celsius << " degrees Celsius are "
            << 9 * celsius / 5 + 32 << " degrees Fahrenheit.\n";
  return 0;
}
```

Program 5: *progs/fahrenheit.C*

If you try out the program on the input of 15 degrees Celsius, you will get the following output.

```
15 degrees Celsius are 59 degrees Fahrenheit.
```

This output is produced when the expression statement in lines 14–15 of the program is executed. Here we focus on the evaluation of the arithmetic expression

```
9 * celsius / 5 + 32
```

in line 15. This expression contains the primary expressions 9, 5, 32, and `celsius`, where `celsius` is a variable of type `int`. This fundamental type is one of the *arithmetic types* in C++.

Literals of type int. 9, 5 and 32 are decimal literals of type `int`, with their values immediately apparent. Decimal literals of type `int` consist of a sequence of digits from 0 to 9, where the first digit must not be 0. The value of a decimal literal is the decimal number represented by the sequence of digits. There are no literals for negative integers. You can get value -9 by writing `-9`, but this is a composite expression built from the unary subtraction operator (Section 2.2.4) and the literal 9.

2.2.1 Associativity and precedence of operators

The evaluation of an expression is to a large extent governed by the *associativities* and *precedences* of the involved operators. In short, associativities and precedences determine the logical parentheses in an expression that is not, or only incompletely, parenthesized. We have already touched associativity in connection with the output operator in Section 2.1.13.

C++ allows incompletely parenthesized expressions in order to save parentheses at obvious places. This is like in mathematics, where we write $3 + 4 \cdot 5$ when we mean $3+(4\cdot5)$. We also write $3+4+5$, even though it is not a priori clear whether this means $(3+4)+5$ or $3+(4+5)$. Here, the justification is that addition is *associative*, so it does not matter which variant we mean.

The price to pay for less parentheses is that we have to know the *logical* parentheses. But this is a moderate price, since the two rules that are used most frequently are quite intuitive and easy to remember. Also, there is always the option of explicitly adding parentheses in case you are not sure where C++ would put them. Let us start with the two essential rules for arithmetic expressions.

> **Arithmetic Evaluation Rule 1:** Multiplicative operators have higher precedence than additive operators.

The expression `9 * celsius / 5 + 32` involves the multiplication operator `*`, the division operator `/`, and the addition operator `+`. All three are binary operators. In C++ as in mathematics, the multiplicative operators `*` and `/` have *higher* precedence than the additive operators `+` and `-`. We also say that multiplicative operators *bind* more strongly than additive ones.[4] This means, our expression contains the logical paren-

[4] In American English, this rule is known as "PEMDAS", in British English it's "BODMAS", and in German it's "Punkt- vor Strichrechnung".

theses `(9 * celsius / 5) + 32`: it is a composite expression built from the addition operator and its operands `9 * celsius / 5` and `32`.

Arithmetic Evaluation Rule 2: Binary arithmetic operators are left associative.

In mathematics, it does not matter how the sub-expression `9 * celsius / 5` is parenthesized. But in C++, it is done from *left to right*, that is, the two leftmost sub-expressions are grouped together. This is a consequence of the fact that the binary arithmetic operators are defined to be *left* associative. The expression `9 * celsius / 5` is therefore logically parenthesized as `(9 * celsius) / 5`, and our original expression has to be read as

```
((9 * celsius) / 5) + 32
```

Identifying the operators in an expression. There is one issue we haven't discussed yet, namely that *different* C++ operators may have the *same* token. For example, `-` can be a binary operator as in `3 - 4`, but it can also be a unary operator as in `-5`. Which one is meant must be inferred from the context. Usually, this is clear, and in cases where it is not (but also in other cases), it is probably a good idea to add some extra parentheses to make the expression more readable (see also the Details section below).

Let us consider another concrete example, the expression `-3 - 4`. It is clear that the first `-` must be unary (there is no left hand side operand), while the second one is binary (there are operands on both sides). But is this expression logically parenthesized as `-(3 - 4)`, or as `(-3) - 4`? Since we get different values in both cases, we better make sure that we know the answer.

The correct logical parentheses are

$$((-3)-4),$$

so the value of the expression `-3 - 4` is -7. This follows from the third most important rule for arithmetic expressions.

Arithmetic Evaluation Rule 3: Unary operators + and - have higher precedence than their binary counterparts.

By using (explicit) parentheses as in `9 * (celsius + 5) * 32`, precedences can be overruled. To get the logical parentheses for such a partially parenthesized expression, we apply the rules from above, considering the already parenthesized parts as operands. In the example, this leads to the logical parentheses `(9 * (celsius + 5)) * 32`.

The Details section discusses how to parenthesize a general expression involving arbitrary C++ operators, using their arities, precedences and associativities.

2.2.2 Expression trees

In any composite expression, the logical parentheses determine a unique “top-level” operator, namely the one that appears within a smallest number of parentheses. The expression is then a composite expression, built from the top-level operator and its operands that are again expressions.

The recursive structure of an expression can nicely be visualized in the form of an *expression tree*. In Figure 3, the expression tree for the expression `9 * celsius / 5 + 32` is shown.

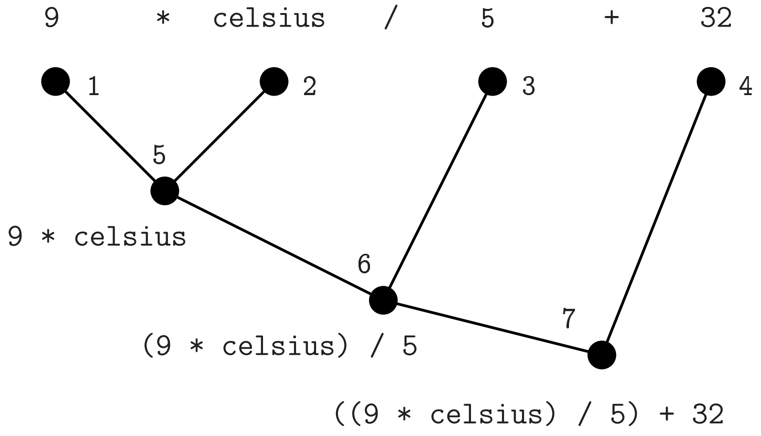


Figure 3: *An expression tree for* `9 * celsius / 5 + 32` *and its logical parentheses* `((9 * celsius) / 5) + 32`. *Nodes are labeled from one to seven.*

How do we get this tree? The expression itself defines the *root* of the tree, and the operands of the top-level operator become the root’s *children* in the tree. Each operand then serves as the root of another subtree. When we reach a primary expression, it defines a *leaf* in the tree, with no further children.

2.2.3 Evaluating expressions

From an expression tree we can easily read off the possible *evaluation sequences* for the arithmetic expression. Such a sequence contains all sub-expressions occurring in the tree, ordered by their time of evaluation. For this sequence to be valid, we have to make sure that we evaluate an expression only *after* the expressions corresponding to *all* its children have been evaluated. By looking at Figure 3, this becomes clear: before evaluating `9 * celsius`, we have to evaluate `9` and `celsius`, otherwise, we don’t have enough information to perform the evaluation.

When we associate the evaluation sequence with the corresponding sequence of nodes in the tree, a valid node sequence *topologically sorts* the tree. This means that any node in the sequence occurs only after all its children have occurred. In Figure 3, for example, the node sequence $(1, 2, 5, 3, 6, 4, 7)$ induces a valid evaluation sequence. Assuming that the variable celsius has value 15, we obtain the following evaluation sequence. (In each step, the sub-expression to be evaluated next is marked by a surrounding box.)

$$\begin{array}{lll}
\boxed{\texttt{9}}\ \texttt{* celsius / 5 + 32} & \longrightarrow^1 & 9 * \boxed{\texttt{celsius}} / 5 + 32 \\
& \longrightarrow^2 & \boxed{\texttt{9 * 15}} / 5 + 32 \\
& \longrightarrow^5 & 135 / \boxed{5} + 32 \\
& \longrightarrow^3 & \boxed{\texttt{135 / 5}} + 32 \\
& \longrightarrow^6 & 27 + \boxed{32} \\
& \longrightarrow^4 & \boxed{\texttt{27 + 32}} \\
& \longrightarrow^7 & 59
\end{array}$$

The sequence $(1, 2, 3, 4, 5, 6, 7)$ is another valid node sequence, inducing a different evaluation sequence; the resulting value of 59 is the same. There are much more evaluation sequences, of course, and it is unspecified by the C++ standard which one is to be used.

In our small example, all possible evaluation sequences will result in value 59, but it is also not hard to write down expressions whose values and effects depend on the evaluation sequence being chosen (see Exercise 2*(g)*, Exercise 13*(h)*, and the Details section below). A program that contains such an expression might exhibit unspecified behavior. But through good programming style, this issue is easy to avoid, since it typically only occurs when one tries to squeeze too much functionality into a single line of code.

2.2.4 Arithmetic operators on the type int

In the program fahrenheit.C, we have already encountered the multiplicative operators * and /, as well as the binary addition operator +. Its obvious counterpart is the binary subtraction operator -.

Table 1 lists arithmetic operators (and the derived *assignment operators*) that are available for the type int, with their arities, precedences and associativities. The actual numbers that appear in the precedence column are not relevant: it is the *order* among precedences that matters.

Let us discuss the functionalities of these operators in turn, where *, + and - are self-explanatory. But already the division operator requires a discussion.

The division operator. According to the rules of mathematics, we could replace the expression

```
9 * celsius / 5 + 32
```

by the expression

```
9 / 5 * celsius + 32
```

Description	Operator	Arity	Prec.	Assoc.
post-increment	++	1	17	left
post-decrement	--	1	17	left
pre-increment	++	1	16	right
pre-decrement	--	1	16	right
sign	+	1	16	right
sign	-	1	16	right
multiplication	*	2	14	left
division (integer)	/	2	14	left
modulus	%	2	14	left
addition	+	2	13	left
subtraction	-	2	13	left
assignment	=	2	4	right
mult assignment	*=	2	4	right
div assignment	/=	2	4	right
mod assignment	%=	2	4	right
add assignment	+=	2	4	right
sub assignment	-=	2	4	right

Table 1: *Arithmetic and assignment operators for the type* `int`. *Each increment or decrement operator expects an lvalue. The composite expression is an lvalue (pre-increment and pre-decrement), or an rvalue (post-increment and post-decrement). Each assignment operator expects an lvalue as left operand and an rvalue as right operand; the composite expression is an lvalue. All other operators involve rvalues only and have no effects.*

without affecting its value and the functionality of the program `fahrenheit.C`. But if we run the program with the latter version of the expression on the input of 15 degrees Celsius, we get the following output:

```
15 degrees Celsius are 47 degrees Fahrenheit.
```

This result is fairly different from our previous (and correct) result of 59 degrees Fahrenheit, so what is going on here? The answer is that the binary division operator `/` on the type `int` implements the *integer division*, in mathematics denoted by div. This does not correspond to the regular division where the quotient of two integers is in general a non-integral rational number.

The modulus operator. The remainder of the integer division can be obtained with the binary *modulus* operator `%`, in mathematics denoted by mod. The mathematical rule

$$a = (a \operatorname{div} b)b + a \bmod b$$

also holds in C++: for example, if `a` and `b` are variables of type `int`, the value of `b` being non-zero, the expression

```
(a / b) * b + a % b
```

has the same value as `a`. The modulus operator is considered as a multiplicative operator and has the same precedence (14) and associativity (left) as the other two multiplicative operators `*` and `/`.

If both `a` and `b` have non-negative values, then `a % b` has a non-negative value as well. This implies that the integer division rounds *down* in this case. If (at least) one of `a` or `b` has a negative value, it is implementation defined whether division rounds up or down.[5] Note that by the identity `(a / b) * b + a % b`, the rounding mode for division also determines the functionality of the modulus operator. If `b` has value 0, the values of `a / b` and `a % b` are undefined.

Coming back to our example (and taking precedences and associativities into account), we get the following valid evaluation sequence for our alternative Celsius-to-Fahrenheit conversion.[6]

```
9 / 5 * celsius + 32 ⟶ 1 * celsius + 32
                     ⟶ 1 * 15 + 32
                     ⟶ 15 + 32
                     ⟶ 47
```

Here we see the "error" made by the integer division: `9 / 5` has value 1.

[5] There is a remark in the standard that future revisions may prescribe a rounding towards zero for these cases.

[6] To avoid longish evaluation sequences, we will from now on suppress the evaluation of literals.

Unary additive operators. We have already touched the unary - operator, and this operator does what one expects: the value of the composite expression -*expr* is the negative of the value of *expr*. There is a unary + operator, for completeness, although its "functionality" is non-existing: the value of the composite expression +*expr* is the same as the value of *expr*.

Increment and decrement operators. Each of the tokens ++ and -- is associated with two *distinct* unary operators that differ in precedence and associativity.

The pre-increment ++ and the pre-decrement -- are right associative. The effect of the composite expressions ++*expr* and --*expr* is to increase (decrease, respectively) the value of *expr* by 1. Then, the object referred to by *expr* is returned. For this to make sense, *expr* has to be an lvalue. We also say that pre-increment is ++ in *prefix notation*, and similarly for --.

The post-increment ++ and the post-decrement -- are left associative. As before, the effect of the composite expressions *expr*++ and *expr*-- is to increase (respectively decrease) the value of *expr* by 1, and *expr* has to be an lvalue for this to work. The return value, though, is an rvalue corresponding to the *old* value of *expr before* the increment or decrement took place. We also say that post-increment is ++ in *postfix notation*, and similarly for --.

The difference between the increment operators in pre- and postfix notation is illustrated in the following example program.

```
#include <iostream>
int main() {
  int a = 7;
  std::cout << ++a << "\n"; // outputs 8
  std::cout << a++ << "\n"; // outputs 8
  std::cout << a   << "\n"; // outputs 9
  return 0;
}
```

You may argue that the increment and decrement operators are superfluous, since their functionality can be realized by combining the assignment operator (Section 2.1.13) with an additive operator. Indeed, if a is a variable, the expression ++a is equivalent in value and effect to the expression `a = a + 1`. There is one subtlety, though: if *expr* is a general lvalue, ++*expr* is *not* necessarily equivalent to *expr* = *expr* + 1. The reason is that in the former expression, *expr* is evaluated only once, while in the latter, it is evaluated *twice*. If *expr* has an effect, this can make a difference.

On the other hand, this subtlety is not the reason why increment and decrement operators are so popular and widely used in C++. After all, it *would* be easy to avoid them in practice. The truth is that incrementing or decrementing values by 1 are such frequent operations in typical C++ code that it pays off to have shortcuts for them.

Prefer pre-increment over post-increment. The statements `++i;` and `i++;` are obviously equivalent, as their effect is the same and the value of the expression is not used. You can exchange them with each other arbitrarily without affecting the behavior of the surrounding program. Whenever you have this choice, you should opt for the pre-increment operator. Pre-increment is the simpler operation because the value of `++i` can simply be read off the variable `i`. In contrast, the post-increment has to "remember" the original value of `i`. As pre-increment is simpler, it also tends to be more efficient.

Remark: We write "pre-increment tends to be more efficient" because in many cases the compiler realizes when the value of an expression is not used. In such a case, the compiler may choose on its own to replace the post-increment in the source code by a "pre-increment" in machine language as an optimization. However, there is absolutely no benefit in choosing a post-increment where a pre-increment would do as well. In this case, you should take the burden from the compiler and optimize by yourself.

Also, post-increment and post-decrement are the only unary C++ operators that are left associative. This makes their usage appear somewhat counterintuitive.

Assignment operators. The assignment operator `=` is available for all types, see Section 2.1.13. But there are specific operators that combine the arithmetic operators with an assignment. These are the binary operators `+=`, `-=`, `*=`, `/=` and `%=`. The expression *expr1* `+=` *expr2* has the effect of adding the value of *expr2* (an rvalue) to the value of *expr1* (an lvalue). The object referred to by *expr1* is returned. This is a generalization of the pre-increment: the expression `++`*expr* is equivalent to *expr* `+= 1`. As before, *expr1* `+=` *expr2* is *not* equivalent to *expr1* `=` *expr1* `+` *expr2* in general, since the latter expression evaluates *expr1* twice.

The operators `-=`, `*=`, `/=` and `%=` work in the same fashion, based on the subtraction, multiplication, division, and modulus operator, respectively.

All the assignment operators have precedence 4, i.e. they bind more weakly than the other arithmetic operators. This is quite intuitive: `a=b*c-d`, say, means `a=(b*c-d)`.

2.2.5 Value range

A variable of type `int` is associated with a *fixed* number of memory cells, and therefore also with a fixed number of bits, say `b`. We call this a `b`-*bit representation*.

Such a representation implies that an object of type `int` can assume only finitely many different values. Since any bit can independently have two states, the maximum number of representable values is 2^b, and the actual value range is defined as the set

$$\{-2^{b-1}, -2^{b-1}+1, \ldots, -1, 0, 1, \ldots, 2^{b-1}-1\} \subset \mathbb{Z}$$

of 2^b numbers.[7] You can find out the smallest and largest `int` values on your computer, using the library `limits`. The corresponding code is given in Program 6.

[7] The C++ standard does not prescribe this, but any different choice of value range would be somewhat unreasonable, given other requirements imposed by the standard.

```
// Program: limits.C
// Output the smallest and the largest value of type int.

#include <iostream>
#include <limits>

int main()
{
  std::cout << "Minimum int value is "
            << std::numeric_limits<int>::min() << ".\n"
            << "Maximum int value is "
            << std::numeric_limits<int>::max() << ".\n";
  return 0;
}
```

Program 6: *progs/limits.C*

When you run the program `limits.C` on a 32-bit system, you will most likely get the following output.

```
Minimum int value is -2147483648.
Maximum int value is 2147483647.
```

Indeed, as $2147483647 = 2^{31}-1$, you can deduce that the number of bits used to represent an `int` value on this system is 32. At this point, you are not supposed to understand the expression `std::numeric_limits<int>::min()` in detail, but we believe that you get its idea.

It is clear that the arithmetic operators (except the unary + and the binary / and %) cannot work exactly like their mathematical counterparts, even when their arguments are restricted to representable `int` values. The reason is that the values of composite expressions constructed from these operators can under- or overflow the value range of the type `int`. The most obvious such example is the expression `2147483647+1`. As we have just seen, its mathematically correct value of 2147483648 is not representable over the type `int` on your system, so you will inevitably get some other value.

Such under- and overflows are a severe problem in many practical applications, but it would be an even more severe problem not to know that they can occur.

2.2.6 The type unsigned int

An object of type `int` can have negative values, but often we only work with natural numbers.[8] Using a type that represents only non-negative values allows to extend the range of positive values without using more bits. C++ provides such a type, it is called `unsigned int`. On this type, we have all the arithmetic operators we also have for `int`,

[8] For us, the set $\mathbb{N}$ of natural numbers starts with 0, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

with the same arities, precedences and associativities. Given a b-bit representation, the value range of `unsigned int` is the set

$$\{0, 1, \ldots, 2^b - 1\} \subset \mathbb{N}$$

of 2^b natural numbers. Indeed, when you replace all occurrences of `int` by `unsigned int` in the program `limits.C`, it may produce the following output.

```
Minimum value of an unsigned int object is 0.
Maximum value of an unsigned int object is 4294967295.
```

Literals of type `unsigned int` look like literals of type `int`, followed by either the letter `u` or `U`. For example, `127u` and `0U` are literals of type `unsigned int`, with their values immediately apparent.

2.2.7 Mixed expressions and conversions

Expressions may involve sub-expressions of type `int` *and* of type `unsigned int`. For example `17+17u` is a legal arithmetic expression, but what are its type and value? In such *mixed expressions*, the operands are implicitly *converted* to the *more general* type. By the C++ standard, the more general type is `unsigned int`. Therefore, the expression `17+17u` is of type `unsigned int` and gets evaluated step by step as

`17+17u` $\longrightarrow$ `17u+17u` $\longrightarrow$ `34u`

This might be somewhat confusing, since in mathematics, it is just the other way around: $\mathbb{Z}$ (the set of integers) is more general than $\mathbb{N}$ (the set of natural numbers). We are not aware of any deeper justification for the way it is done in C++, but at least the conversion is well-defined:

Non-negative `int` values are "converted" to the same value of type `unsigned int`; negative `int` values are converted to the `unsigned int` value that results from (mathematically) adding 2^b. This rule establishes a bijection between the value ranges of `int` and `unsigned int`.

Implicit conversions in the other direction may also occur but are not always well-defined. Consider for example the declarations

```
int a = 3u;
int b = 4294967295u;
```

The value of `a` is 3, since this value is in the range of the type `int`. But if we assume the 32-bit system from above, the value of `b` is implementation defined according to the C++ standard, since the literal `4294967295` is outside the range of `int`.

2.2.8 Binary representation

Assuming b-bit representation, we already know that the type int covers the values

$$-2^{b-1}, \ldots, 2^{b-1} - 1,$$

while unsigned int covers

$$0, \ldots, 2^b - 1.$$

In this subsection, we want to take a closer look at how these values are represented in memory, using the b available bits. This will also shed more light on some of the material in the previous subsection.

The *binary expansion* of a natural number $n \in \mathbb{N}$ is the sum

$$n = \sum_{i=0}^{\infty} b_i 2^i,$$

where the b_i are uniquely determined coefficients from $\{0, 1\}$, with only finitely many of them being nonzero. For example,

$$13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3.$$

The sequence of the b_i in reverse order is called the binary representation of n. The binary representation of 13 is 1101, for example.

Conversion decimal $\to$ binary. The identity

$$n = \sum_{i=0}^{\infty} b_i 2^i = b_0 + \sum_{i=1}^{\infty} b_i 2^i = b_0 + \sum_{i=0}^{\infty} b_{i+1} 2^{i+1} = b_0 + 2 \underbrace{\sum_{i=0}^{\infty} b_{i+1} 2^i}_{=:n'}$$

provides a simple algorithm to compute the binary representation of a given decimal number $n \in \mathbb{N}$. The least significant coefficient b_0 of the binary expansion of n is $n \bmod 2$. The other coefficients b_i, $i \geq 1$, can subsequently be extracted by applying the same technique to $n' = (n - b_0)/2$.

For example, for $n = 14$ we get $b_0 = 14 \bmod 2 = 0$ and $n' = (14 - 0)/2 = 7$. We continue with $n = 7$ and get $b_1 = 7 \bmod 2 = 1$ and $n' = (7 - 1)/2 = 3$. For $n = 3$ we get $b_2 = 3 \bmod 2 = 1$ and $n' = (3 - 1)/2 = 1$ which leaves us with $n = b_3 = 1$. In summary, the binary representation of 14 is $b_3 b_2 b_1 b_0 = 1110$.

Conversion binary $\to$ decimal. To convert a given binary number $b_k \ldots b_0$ into decimal representation, we can once again use the identity from above.

$$\sum_{i=0}^{k} b_i 2^i = b_0 + 2 \sum_{i=0}^{k-1} b_{i+1} 2^i = \ldots = b_0 + 2(b_1 + 2(b_2 + 2(\cdots + 2b_k) \ldots))$$

For example, to convert the binary number $b_4b_3b_2b_1b_0 = 10100$ into decimal representation, we compute

$$(((b_4 \cdot 2 + b_3) \cdot 2 + b_2) \cdot 2 + b_1) \cdot 2 + b_0 \;=\; (((1 \cdot 2 + 0) \cdot 2 + 1) \cdot 2 + 0) \cdot 2 + 0 \;=\; 20.$$

Representing unsigned int values. Since any `unsigned int` value

$$n \in \{0, \ldots, 2^b - 1\}$$

has a binary representation of length exactly b (filling up with leading zeros), this binary representation is a canonical format for storing n using the b available bits. Like the value range itself, this storage format is not explicitly prescribed by the C++ standard, but hardly anything else makes sense in practice. As there are 2^b `unsigned int` values, and the same number of b-bit patterns, each pattern encodes one value. For $b = 3$, this looks as follows.

n	representation
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Representing int values. A common way of representing `int` values using the same b bits goes as follows. If the value n is non-negative, we store the binary representation of n itself—a number from

$$\{0, \ldots, 2^{b-1} - 1\}.$$

That way we use all the b-bit patterns that start with 0.

If the value n is negative, we store the binary representation of $n + 2^b$, a number from

$$\{2^{b-1}, \ldots, 2^b - 1\}.$$

This yields the missing b-bit patterns, the ones that start with 1. For $b = 3$, the resulting representations are

n	representation
-4	100
-3	101
-2	110
-1	111
0	000
1	001
2	010
3	011

This is called the *two's complement* representation. In this representation, adding two int values n and n' is very easy: simply add the representations according to the usual rules of binary number addition, and ignore the overflow bit (if any). For example, to add -2 and -1 in case of $b = 3$, we compute

$$\begin{array}{rr} & 110 \\ + & 111 \\ \hline & 1101 \end{array}$$

Ignoring the leftmost overflow bit, this gives 101, the representation of the result -3 in two's complement. This works since the binary number behind the encoding of n is either n or $n + 2^b$. Thus, when we add the binary numbers for n and n', the result is congruent to $n + n'$ modulo 2^b and therefore agrees with $n + n'$ in the b rightmost bits.

Using the two's complement representation we can now better understand what happens when a negative int value n gets converted to type unsigned int. The standard specifies that for this, n has to be incremented by 2^b. But under the two's complement, the negative int value n and the resulting positive unsigned int value $n + 2^b$ have the same representation! This means that the conversion is purely conceptual, and no actual computation takes place.

The C++ standard does not prescribe the use of the two's complement, but the rule for conversion from int to unsigned int is clearly motivated by it.

2.2.9 Integral types

There is a number of other fundamental types to represent signed and unsigned integers, see the Details section. These types may differ from int and unsigned int with respect to their value range. All these types are called *integral types*, and for each of them, all the operators in Table 1 (Page 48) are available, with the same arities, precedences, associativities and functionalities (up to the obvious limits dictated by the respective value ranges).

2.2.10 Details

Literals. There are also non-decimal literals of type `int`. An *octal* literal starts with the digit `0`, followed by a sequence of digits from `0` to `7`. The value is the octal number represented by the sequence of digits following the leading `0`. For example, the literal `011` has value $9 = 1 \cdot 8^1 + 1 \cdot 8^0$.

Hexadecimal literals start with `0x`, followed by a sequence of digits from `0` to `9` and letters from `A` to `F`. The value is the hexadecimal number represented by the sequence of digits and letters following the leading `0x`. For example, the literal `0x1F` has value $31 = 1 \cdot 16^1 + 15 \cdot 16^0$.

Logically parenthesizing a general expression. Given an expression that consists of a sequence of operators and operands, we want to deduce the logical parentheses. For each operator in the sequence, we know its arity, its precedence (a number between 1 and 18, see Table 1 on Page 48 for the arithmetic operators), and its associativity (left or right). In case of a unary operator, the associativity specifies on which side of the operator its operand is to be found.

Let us consider the following abstract example to emphasize that what we do here is completely general and not restricted to arithmetic expressions.

expression	x_1	op_1	x_2	op_2	x_3	op_3	op_4	x_4
arity		2		2		2	1	
precedence		4		13		13	16	
associativity		r		l		l	r	

Here is how the parentheses are obtained: for each operator, we identify its *leading* operand, defined as the left hand side operand for left associative operators, and as the right hand side operand otherwise. The leading operand for op_i includes everything to the relevant side between op_i and the next operator of *lower* precedence than op_i. In other words, everything in between these two operators is "grabbed" by the "stronger" operator.

In our example, the leading operand of op_3 is the subsequence $x_2 \ op_2 \ x_3$ to the left of op_3, since the next operator of lower precedence to the left of op_3 is op_1.

In the case of binary operators, we also find the *secondary* operand, the one to the other side of the leading operand. The secondary operand for op_i includes everything to the relevant side between op_i and the next operator of *the same or lower* precedence than op_i. The only difference to the leading operand rule is that the secondary operand already ends when an operator of the *same* precedence appears.

According to this definition, the secondary operand of op_3 is $op_4 \ x_4$ in our example.

Finally, we put a pair of parentheses around the subsequence corresponding to the leading operand, the operator itself, and the secondary operand (if any).

Here is the table for our example again, enhanced with the subsequences of all four operators that are put in parentheses according to the rules just described.

expression	x_1	op_1	x_2	op_2	x_3	op_3	op_4	x_4
arity		2		2		2	1	
precedence		4		13		13	16	
associativity		r		l		l	r	
op_1	(x_1	op_1	x_2	op_2	x_3	op_3	op_4	x_4)
op_2			(x_2	op_2	x_3)			
op_3			(x_2	op_2	x_3	op_3	op_4	x_4)
op_4							(op_4	x_4)

Now we simply put together all parentheses that we have obtained, taking their multiplicities into account. In our example we get the expression

$$(\ x_1\ op_1\ ((\ x_2\ op_2\ x_3\ )\ op_3\ (op_4\ x_4\ ))).$$

By some magic, this worked out, and we have a fully parenthesized expression (the outer pair of parentheses can be dropped again, of course). But note that we cannot expect such nice behavior in general. Consider the following example.

expression	x_1	op_1	x_2	op_2	x_3
arity		2		2	
precedence		13		13	
associativity		r		l	
op_1	(x_1	op_1	x_2	op_2	x_3)
op_2	(x_1	op_1	x_2	op_2	x_3)

The resulting parenthesized expression is

$$((\ x_1\ op_1\ x_2\ op_2\ x_3\ )),$$

which does not specify the evaluation order. What comes to our rescue is that C++ only allows expressions for which the magic works out! The previous bad case is impossible, for example, since all binary operators of the same precedence also have the same associativity.

Unsigned arithmetic. We have discussed how `int` values are converted to `unsigned int` values, and vice versa. The main issue (what to do with non-representable values) also occurs during evaluation of arithmetic expressions involving only *one* of the types. The C++ standard contains one rule for this. For all unsigned integral types, the arithmetic operators work modulo 2^b, given b-bit representation. This means that the value of any arithmetic operation with operands of type `unsigned int` is well-defined. It does not necessarily give the mathematically correct value, but the unique value in the `unsigned int` range that is congruent to it modulo 2^b. For example, if `a` is a variable of type `unsigned int` with non-zero value, then `-a` has value $2^b - a$.

No such rule exists for the signed integral types, meaning that over- and underflow are dealt with at the discretion of the compiler.

Sequences of + and -. We have argued above that it is usually clear which operators occur in an expression, even though some of them share their token. But since the characters + and - are heavily overused in operator tokens, special rules are needed to resolve the meanings of sequences of +'s, or of -'s.

For example, only from arities, precedences and associativities it is not clear how to interpret the expressions `a+++b` or `---a`. The first expression could mean `(a++)+b`, but it could as well mean `a+(++b)` or `a+(+(+b)`. Similarly, the second expression could either mean `-(--a)`, `--(-a)` or `-(-(-a)`.

The C++ standard resolves this dilemma by defining that a sequence consisting only of +'s, or only of -'s, has to be grouped into pairs from left to right, with possibly one remaining + or - at the end. Thus, `a+++b` means `(a++)+b`, and `---a` means `--(-a)`. Note that for example the expression `a++b` *would* make sense when parenthesized as `a+(+b)`, but according to the rule just established, it is not a well-formed expression, since a unary operator `++` cannot have operands on both sides. The expression `---a` with its logical parentheses `--(-a)` is invalid for another reason: the operand of the pre-increment must be an lvalue, but the expression `-a` is an rvalue.

Other integral types. C++ contains a number of fundamental *signed* and *unsigned* integral types. The signed ones are `signed char`, `short int`, `int` and `long int`. The standard specifies that each of them is represented by at least as many bits as the previous one in the list. The number of bits used to represent `int` values depends on the platform. The corresponding sequence of unsigned types is `unsigned char`, `unsigned short int`, `unsigned int` and `unsigned long int`.

These types give compilers the freedom of offering integers with larger or smaller value ranges than `int` and `unsigned int`. Smaller value ranges are useful when memory consumption is a concern, and larger ones are attractive when over- and underflow occurs. The significance of these types (which are already present in the C programming language) has faded in C++. The reason is that we can quite easily implement our own tailor-made integral types in C++, if we need them. In C this is much more cumbersome. Consequently, many C++ compilers simply make `short int` and `long int` an alias for `int`, and the same holds for the corresponding unsigned types.

Order of effects and sequence points Increment and decrement operators as well as assignment operators construct expressions with an effect. Such operators have to be used with care for two reasons.

The obvious reason is that (as we already learned in the end of Section 2.1.1) the evaluation order for the sub-expressions of a given expression is not specified in general. Consequently, value and effect may depend on the evaluation order. Consider the expression

```
++i + i
```

where we suppose that `i` is a variable of type `int`. If `i` is initially 5, say, then the value of the composite expression may in practice be 11 or 12. The result depends on whether or not the effect of the left operand `++i` of the addition is processed before the right operand `i` is evaluated. The value of the expression `++i + i` is therefore unspecified by the C++ standard.

To explain the second (and much less obvious, but fortunately also much less relevant) reason, let us consider the following innocent looking expression that involves a variable `i` of type `int`.

```
i = ++i + 1
```

This expression has two effects: the increment of `i` and the assignment to `i`. Because the assignment can only happen after the operands have been evaluated, it seems that the order of the two effects is clear: the increment comes before the assignment, and the overall value and effect are well-defined.

However, this is not true, for reasons that have to do with our underlying computer model, the von Neumann architecture. From the computer's point of view, the evaluation of the sub-expression `++i` consists of the following steps.

1. Copy the value of `i` from the main memory into one of the CPU registers;

2. Add 1 to this value in the register;

3. Write the register content back to main memory, at the address of `i`;

Clearly, the first two steps are necessary to obtain the value of the expression `++i` and, hence, have to be processed before the assignment. But the third step does not necessarily have to be completed before the assignment. In order to allow the compiler to optimize the transfer of data between CPU registers and main memory (which is very much platform dependent), this order has not been specified. In fact, it is not unreasonable to assume that the traffic between registers and main memory is organized such that several items are transfered at once or quickly after another, using so-called *bursts*.

Suppose as before that `i` initially has value 5. If the assignment is performed after the register content is written back to main memory, `i = ++i + 1` sets `i` to 7. But if the assignment happens *before*, the later transfer of the register value 6 overrides the previous value of 7, and `i` is set to 6 instead.

The C++standard defines a *sequence point* to be a point during the evaluation sequence of an expression at which is guaranteed that all effects of previously evaluated (sub)expressions have been carried out. It was probably the existence of highly optimized C compilers that let the C++ standard refrain from declaring the assignment as a sequence point. In other words, when the assignment to `i` takes place in the evaluation `i = ++i + 1`, it is not specified whether the effect of the previously evaluated increment operator has been carried out or not. In contrast, the semicolon that terminates an expression statement is always a sequence point.

Therefore, we only have an issue with expressions that have more than one effect. Hence, if you prefer not to worry about effect order, ensure that each expression that you write generates at most one effect. Expressions with more than one effect can make sense, though, and they are ok, as long as some sequence points separate the effects and put them into a well-defined order. This is summarized in the following rule.

Single Modification Rule: Between two sequence points, the evaluation of an expression may modify the value of an object of fundamental type at most once.

An expression like `i = ++i + 1` that violates this rule is considered semantically illegal and leads to undefined behavior.

If you perceive this example as artificial, here is a "more natural" violation of the single modification rule: if `nextvalue` is a variable of type `int`, it might seem that

```
nextvalue = 5 * nextvalue + 3
```

could more compactly be written as

```
(nextvalue *= 5) += 3
```

This will compile: `(nextvalue *= 5)` is an lvalue, so we can assign to it. Still, the latter expression is invalid since it modifies `nextvalue` twice.

At this point, an attentive reader should wonder how an expression that involves several output operators complies with the Single Modification Rule. Indeed, an expression like

```
std::cout << a << "^8 = " << b * b << ".\n"
```

has several effects all of which modify the lvalue `std::cout`. This works since the type of `std::cout` (which we will not discuss here) is *not* fundamental and, hence, the Single Modification Rule does not apply in this case.

2.2.11 Goals

Dispositional. At this point, you should ...

1) know the three Arithmetic Evaluation Rules;
2) understand the concepts of operator precedence and associativity;
3) know the arithmetic operators for the types `int` and `unsigned int`;
4) be aware that computations involving the types `int` and `unsigned int` may deliver incorrect results, due to possible over- and underflows.

Operational. In particular, you should be able to ...

(G1) parenthesize and evaluate a given arithmetic expression involving operands of types `unsigned int` and `int`, the binary arithmetic operators +,-, *, /, %, and the unary - (the paragraph on parenthesizing a general expression in the Details section enables you to do this for all arithmetic operators);

(G2) derive basic statements about arithmetic expressions;

(G3) convert a given decimal number into binary representation and vice versa;

(G4) derive the *two's complement* representation of a given number in b-bit representation, for some $b \in \mathbb{N}$;

(G5) write programs whose output is determined by a fixed number of arithmetic expressions involving literals and input variables of types `int` and `unsigned int`;

(G6) determine the value range of integral types on a given machine (using a program).

2.2.12 Exercises

Exercise 11 *Parenthesize the following expressions and then evaluate them step by step. This means that types and values of all intermediate results that are computed during the evaluation should be provided.* (G1)

a)	`-2-4*3`	*b)*	`10%6*8%3`	*c)*	`6-3+4*5`
d)	`5u+5*3u`	*e)*	`31/4/2`	*f)*	`-1-1u+1-(-1)`

Exercise 12 *Which of the following character sequences are not legal expressions, and why? For the ones that are, give the logical parentheses. (In order to avoid (misleading?) hints, we have removed the spaces that we usually include for the sake of better readability.)* (G1)

a)	`c=a+7+--b`	*b)*	`c=-a=b`	*c)*	`c=a=-b`
d)	`a-a/b*b`	*e)*	`b*=++a+b`	*f)*	`a-a*+-b`
g)	`7+a=b*2`	*h)*	`a+3*--b+a++`	*i)*	`b+++--a`

These exercises require you to read the paragraph on logically parenthesizing a general expression in the Details section. Exercise *i)* also requires you to read the paragraph on sequences of `+` and `-` in the Details section.

Exercise 13 *For all legal expressions from Exercise 12, provide a step-by-step evaluation, supposing that initially* `a` *has value* 5, `b` *has value* 2, *and the value of* `c` *is undefined. Which of the expressions result in unspecified or undefined behavior?* (G1)

Exercise 14 *Prove that for all* $a \geq 0$ *and* $b, c > 0$, *the following equation holds.*

$$(a \operatorname{div} b) \operatorname{div} c = a \operatorname{div}(bc).$$

Does this imply that the two expressions `a/b/c` *and* `a/(b*c)` *are equivalent for all such values of the variables* `a`, `b`, *and* `c` *(which are assumed to be of type* `unsigned int`*)?* (G2)

Exercise 15 *Compute by hand binary representations of the following decimal numbers.* (G3)

a) 15 *b)* 172 *c)* 329 *d)* 1022

Exercise 16 *Compute by hand decimal representations of the following binary numbers.* (G3)
a) 110111 *b)* 1000001 *c)* 11101001 *d)* 101010101

Exercise 17 *By September 2008, the largest known Mersenne Prime is* $2^{43,112,609} - 1$. *What is the number of decimal digits that this number has? Explain how you got your answer!* (G3)

Hint: You may need the basic rules of logarithms and a pocket calculator.

Exercise 18 *Assuming a 4-bit representation, compute the binary two's complement representations of the following decimal numbers.* (G4)
a) 6 *b)* -4 *c)* -8 *d)* 9 *e)* -3

Exercise 19 *Write a program* `celsius.C` *that converts temperatures from degrees Fahrenheit into degrees Celsius.*

The initial output that prompts the user to enter the temperature in degrees Fahrenheit should also contain lower *and* upper *bounds for the allowed inputs. These bounds should be chosen such that no over- and underflows can occur in the subsequent computations, given that the user respects the bounds. You may for this exercise assume that the integer division rounds towards zero for all operands: for example,* `-5 / 2` *then rounds the exact result* -2.5 *to* -2.

The program should output the correct *result in degrees Celsius as a mixed rational number of the form* $x\frac{y}{9}$ *(meaning* $x + y/9$*), where* $x, y \in \mathbb{Z}$ *and* $|y| \leq 8$. *For example,* $13\frac{4}{9}$ *could be output simply as* `13 4/9`. *We also allow for example the output* `-1 -1/9` *(meaning* $-1 - 1/9 = -10/9$*).* (G5)

Exercise 20 *Write a program* `threebin.C` *that reads a (decimal) number* $a \geq 0$ *from standard input and outputs the* last three *bits of* a*'s binary representation. Fill up with leading zeros in case the binary representation has less than three bits.* (G5)

2.2.13 Challenges

Exercise 21 *Josephus was a Jewish military leader in the Jewish-Roman war of 66-73. After the Romans had invaded his garrison town, the few soldiers (among them Josephus) that had survived the killings by the Romans decided to commit suicide. But somehow, Josephus and one of his comrades managed to surrender to the Roman forces without being killed (Josephus later became a Roman citizen and well-known historian).*

This historical event is the background for the Josephus Problem *that offers a (mythical) explanation about how Jospehus was able to avoid suicide. Here is the problem.*

41 people (numbered $0, 1, \ldots, 40$*) are standing in a circle, and every* k*-th person is killed until no one survives. For* $k = 3$*, the killing order is therefore*

$2, 5, 8, \ldots, 38, 0, 4, \ldots$. *Where in the circle does Jospehus have to position himself in order to be the last survivor (who then obviously doesn't need to kill himself)?*

a) *Write a program that solves the Josephus problem; the program should receive as input the number* k *and output the number* $p(k) \in \{0, \ldots, 40\}$ *of the last survivor.*

b) *Let us assume that Josephus is not able to chose his position in the circle, but that he can in return choose the parameter* $k \in \{1, \ldots, 41\}$. *Is it possible for him to survive then, no matter where he initially stands?*

Exercise 22 *A triple* (a, b, c) *of positive integers is called a* Pythagorean triple *if* $a^2 + b^2 = c^2$. *For example,* $(3, 4, 5)$ *is a Pythagorean triple. Write a program* `pythagoras.C` *that allows you to list all Pythagorean triples for which* $a + b + c = 1000$. *We're not demanding that the program lists them directly, but the program should "prove" that your list is correct. How many such Pythagorean triples are there? (This is a slight variation of Problem 9 from the* Project Euler, *see* `http://projecteuler.net/`.*)*

2.3 Booleans

The truth always lies somewhere else.

Unknown

> *This section discusses the type* `bool` *used to represent truth values or Booleans, for short. You will see a number of operations on Booleans and why only few of these operations suffice to express all the others. You will learn how to evaluate expressions involving the type* `bool`*, using* short-circuit evaluation.

What is the simplest C++ type you can think of? If we think of types in terms of their value ranges, then you will probably come up with a type whose value range is empty or consists of one possible value only. Arguably, values of such types are very easy to represent, even without spending any memory resources. However, although such types are useful in certain circumstances, you can't do a lot of interesting computations with them. After all, there is no operation on them other than the identity.

So, let us rephrase the above question: What is the simplest non-trivial C++ type you can think of? After the above discussion we certainly have one candidate: a type with a value range that consists of exactly two elements. At first sight, such a type may again appear very limited. Nevertheless, we will see below that it allows for many interesting operations. Actually, such a type is sufficient as a basis for all kinds of computations you can imagine. (Recall, for example, that integral numbers can be represented in binary format, that is, using the two values 0 and 1 only.)

2.3.1 Boolean functions

The name "Boolean" stems from the British mathematician George Boole (1815–1864) who pioneered in establishing connections between logic and symbolic algebra. By the term *Boolean function* we denote a function $f : \mathcal{B}^n \rightarrow \mathcal{B}$, where $\mathcal{B} := \{0, 1\}$ and $n \in \mathbb{N}$. (Read 0 as *false* and 1 as *true*.)

Clearly the number of different Boolean functions is finite for every fixed n; Exercise 23 asks you to show what exactly their number is. To give you a first hint: For $n = 1$ there are only four Boolean functions, the two constant functions $c_0 : x \mapsto 0$ and $c_1 : x \mapsto 1$, the identity $\mathrm{id} : x \mapsto x$ and the negation $\mathrm{NOT} : x \mapsto \overline{x}$, where $\overline{0} := 1$ and $\overline{1} := 0$.

In the following we restrict our focus to unary and binary Boolean functions, that is, functions from $\mathcal{B}$ or $\mathcal{B}^2$ to $\mathcal{B}$. Such functions are most conveniently described as a small table that lists the function values for all possible arguments. An example for a binary Boolean function is $\mathrm{AND} : (x, y) \mapsto x \wedge y$ shown in Figure 4(a). It is named AND because $x \wedge y = 1$ if and only if $x = 1$ *and* $y = 1$. You may guess why the function $f : (x, y) \mapsto x \vee y$ defined in Figure 4(b) is called OR. In fact, there are two possible interpretations of the word "or": You can read it as "at least one of", but just as well it can mean "either ... or", that is, "exactly one of". The function that corresponds to the latter interpretation is shown in Figure 4(c). It is usually referred to as XOR : $(x, y) \mapsto x \oplus y$ or *exclusive or*. Figure 4(e) depicts the table for the unary function NOT.

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

(a) AND.

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

(b) OR.

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

(c) XOR.

x	y	$x \uparrow y$
0	0	1
0	1	1
1	0	1
1	1	0

(d) NAND.

x	$\overline{x}$
0	1
1	0

(e) NOT.

Figure 4: *Examples for Boolean functions.*

Completeness. Figure 4 shows just a few examples. However, in a certain sense, it shows you everything about binary Boolean functions. Some of these functions are so fundamental that *every* binary Boolean function can be generated from them. For example, XOR can be generated from AND, OR and NOT:

$$\mathrm{XOR}(x, y) = \mathrm{AND}(\mathrm{OR}(x, y), \mathrm{NOT}(\mathrm{AND}(x, y))).$$

Informally, "either or" means "or" but not "and". Formulas like this are easily checked by going through all (four) possible combinations of arguments.

Similarly, the function NAND : $(x, y) \mapsto x \uparrow y$ described in Figure 4(d) can be generated from NOT and AND (hence the name ...):

$$\mathrm{NAND}(x, y) = \mathrm{NOT}(\mathrm{AND}(x, y)).$$

Let us define what we mean by "generate".

Definition 2 *Consider a set* F *of boolean functions. A binary boolean function* f *is called* **generated** *by* F *if* f *can be expressed by a formula that only contains the variables* x *and* y*, the constants* 0 *and* 1*, and the functions from* F.

For a set $\mathcal{F}$ *of binary functions, a set* F *of binary functions is said to be* **complete** *if and only if every function* $f \in \mathcal{F}$ *can be generated by* F.

We are now prepared for a completeness proof.

Theorem 1 *The set of functions* $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NOT}\}$ *is complete for the set of binary Boolean functions.*

Proof. Any binary Boolean function f is completely described by its *characteristic vector* $(f(0,0), f(0,1), f(1,0), f(1,1))$. For example, AND has characteristic vector $(0,0,0,1)$, or 0001 for short. Let $f_{b_1 b_2 b_3 b_4}$ denote the Boolean function with characteristic vector $b_1 b_2 b_3 b_4$. For example, $\text{AND} = f_{0001}$.

In the first step of the proof, we show that all those functions can be generated whose characteristic vector contains a single 1. Indeed,

$$\begin{aligned}
f_{0001}(x,y) &= \text{AND}(x,y), \\
f_{0010}(x,y) &= \text{AND}(x, \text{NOT}(y)), \\
f_{0100}(x,y) &= \text{AND}(y, \text{NOT}(x)), \\
f_{1000}(x,y) &= \text{NOT}(\text{OR}(x,y)).
\end{aligned}$$

To check the formula for f_{0010}, for example, we can create a table for the function $\text{AND}(x, \text{NOT}(y))$ as in Figure 4 and convince ourselves that the resulting characteristic vector is 0010.

In the second step, we show that any function whose characteristic vector is nonzero can be generated. This is done by combining the already generated "single-1" functions through OR, which simply adds up their 1's. For example,

$$\begin{aligned}
f_{1100}(x,y) &= \text{OR}(f_{1000}(x,y), f_{0100}(x,y)), \\
f_{0111}(x,y) &= \text{OR}(\text{OR}(f_{0100}(x,y), f_{0010}(x,y)), f_{0001}(x,y)).
\end{aligned}$$

We abstain from working this argument out formally, since we believe that you get its idea. Finally, we generate f_{0000} as

$$f_{0000}(x,y) = 0.$$

□

Exercise 26 asks you to show that the sets $\{\text{AND}, \text{NOT}\}$, $\{\text{OR}, \text{NOT}\}$, and even the set that consists of the single function $\{\text{NAND}\}$ are complete for the set of binary Boolean functions.

2.3.2 The type bool

In C++, Booleans are represented by the fundamental type `bool`. Its value range consists of the two elements *true* and *false* that are associated with the literals `true` and `false`, respectively. For example,

```
bool b = true;
```

defines a variable `b` of type `bool` and initializes it to *true*.

Formally, the type `bool` is an integral type, defined to be less general than `int` (which in turn is less general than `unsigned int`, see Section 2.2.7).

Logical operators. The complete set of binary Boolean functions is available via the *logical operators* `&&` (AND), `||` (OR), and `!` (NOT). Compared to the notation used in Section 2.3.1 we simply identify 1 with *true* and 0 with *false*. Both `&&` and `||` are binary operators, while `!` is unary. All operands are rvalues of type `bool`, and all logical operators also return rvalues of type `bool`. Like in logics, `&&` binds more strongly than `||`, and `!` binds more strongly than `&&`.[9]

Relational operators. There is also a number of operators on arithmetic types whose result is of type `bool`. For each arithmetic type there exist the six *relational operators* `<`, `>`, `<=`, `>=`, `==`, and `!=`. These are binary operators whose two rvalue operands are of some arithmetic type and whose result is an rvalue of type `bool`. The operators `<=` and `>=` correspond to the mathematical relations $\leq$ and $\geq$, respectively. The operator `==` tests for equality and `!=` tests for inequality.

Since `bool` is an integral type, the relational operators may also have operands of type `bool`. The respective comparisons are done according to the convention *false*<*true*.

> **Watch out!** A frequent beginner's mistake is to use the assignment operator `=` where the equality operator `==` is meant.

As a general rule, arithmetic operators bind more strongly than relational ones, and these in turn bind more strongly than the logical operators.

> **Boolean Evaluation Rule:** Binary arithmetic operators have higher precedence than relational operators, and these have higher precedence than binary logical operators.

For example, the expression

```
7 + x < y && y != 3 * z
```

is logically parenthesized as

```
((7 + x) < y) && (y != (3 * z)).
```

Be careful with mathematical shortcut notation such as $a = b = c$. As a C++ expression,

```
a == b == c
```

is not equivalent to

```
a == b && b == c.
```

By left associativity of `==`, the expression `a == b == c` is logically parenthesized as `(a == b) == c`. If all of `a`, `b`, and `c` are variables of type `int` with value 0, the evaluation yields

`(0 == 0) == 0` $\longrightarrow$ *true* `== 0` $\longrightarrow$ `1 == 0` $\longrightarrow$ *false*,

just the opposite of what you usually mean by $a = b = c$.

[9] Recall that an operator binds more strongly than another if its has higher precedence.

De Morgan's laws. The well-known formulae of how to express AND in terms of OR and vice versa with the help of NOT, are named after the British mathematician Augustus De Morgan (1806–1871). He was a pioneer in symbolic algebra and logics. Also the rigorous formulation of "mathematical induction" as we know and use it today goes back to him. The de-Morgan-formulae state that (in C++-language)

```
!(x && y) == (!x || !y)
```

and

```
!(x || y) == (!x && !y) .
```

These formulae can often be used to transform a *Boolean expression* (an expression of type `bool`) into a "simpler" equivalent form. For example,

```
!(x < y || x + 1 > z) && !(y <= 5 * z || !(y > 7 * z))
```

can equivalently be written as

```
x >= y && x + 1 <= z && y > 5 * z && y > 7 * z
```

which is clearly preferable in terms of readability.

For more details about precedences and associativities of the logical and relational operators, see Table 2. You may find this information helpful in order to solve Exercise 28.

Description	Operator	Arity	Prec.	Assoc.
logical not	!	1	16	right
less	<	2	11	left
greater	>	2	11	left
less or equal	<=	2	11	left
greater or equal	>=	2	11	left
equality	==	2	10	left
inequality	!=	2	10	left
logical and	&&	2	6	left
logical or	||	2	5	left

Table 2: *Precedences and associativities of logical and relational operators. All operands and return values are rvalues.*

Conversion and promotion. It is possible that the two operands of a relational operator have different type. This case is treated in the same way as for the arithmetic operators. The composite expression is evaluated on the more general type, to which the operand of the less general type is implicitly converted. In particular, `bool` operands are converted to the respective integral type of the other operand. Here, the value *false* is converted to 0, and *true* to 1. If the integral type is `int`, this conversion is defined to be a *promotion*. A promotion is a special conversion for which the C++ standard guarantees that no information gets lost.

The conversion goes into the other direction for logical operators. In mixed expressions, the integral operands of logical operators are converted to bool in such a way that 0 is converted to *false* and any other value is converted to *true*.

These conversions also take place in initializations and assignments, as in the following examples.

```
bool b = 5; // b is initialized to true
int i = b;  // i is initialized to 1
```

2.3.3 Short circuit evaluation

The evaluation of expressions involving logical and relational operators proceeds according to the general rules, as discussed in Sections 2.2.1 and 2.2.3. However, there is one important difference regarding the order in which the operands of an operator are evaluated. While in general this order is undefined, the binary logical operators && and || always guarantee that their left operand is evaluated first. Moreover, if the value of the composite expression is already defined by the value of the left operand then the right operand is *not evaluated* at all. This evaluation scheme is known as *short circuit evaluation*.

How can it happen that the final value is already determined by the left operand only? Suppose that in an && operator the left operand evaluates to *false*; then no matter what the right operand gives, the result will always be *false*. Hence, there is no need to evaluate the right operand at all. The analogous situation occurs if in an || operator the left operand evaluates to *true*.

At first sight it looks as if short circuit evaluation is merely a matter of efficiency. But there is another benefit. It occurs when dealing with expressions that are defined for certain parameters only. Consider for example the division operation that is defined for a nonzero divisor only. Due to short circuit evaluation, we can write

```
x != 0 && z / x > y
```

and be sure that this expression is always valid. If the right operand was evaluated for x==0,[10] then the result would be undefined.

2.3.4 Details

Naming. The XOR function is also frequently called *antivalence* and denoted by $\not\leftrightarrow$. The NAND function is also known as *alternate denial* or *Sheffer stroke*. The latter name is after the American mathematician Henry M. Sheffer (1883–1964) who proved that all other logical operations can be expressed in terms of NAND.

[10] having the equality operator, we can now use this as a shortcut for "x is 0"

Bitwise operators. We have seen in Section 2.2.8 that integers can be represented in binary format, that is, as a sequence of bits each of which is either 0 or 1. Boolean functions can naturally be extended to integral types by applying them bitwise to the binary representations.

Definition 3 *Consider a nonnegative integer* b *and two integers* $x = \sum_{i=0}^{b} a_i 2^i$ *and* $y = \sum_{i=0}^{b} b_i 2^i$*, for which* $a_i, b_i \in \{0, 1\}$ *for all* $0 \leq i \leq b$*.*

For a unary Boolean function $f : \{0, 1\} \to \{0, 1\}$ *the* **bitwise operator** φ_f *corresponding to* f *is defined as* $\varphi_f(x) = \sum_{i=0}^{b} f(a_i) 2^i$*.*

For a binary Boolean function $g : \{0, 1\}^2 \to \{0, 1\}$ *the* **bitwise operator** φ_g *corresponding to* g *is defined as* $\varphi_g(x, y) = \sum_{i=0}^{b} g(a_i, b_i) 2^i$*.*

For illustration, suppose that we have an unsigned integral type with a 4-bit representation. That is, 0000 represents 0, 0001 represents 1, and so on, up to 1111 which represents 15.

Then you can check that $\varphi_{\mathrm{OR}}(4, 13) = 13$, $\varphi_{\mathrm{NAND}}(13, 9) = 6$, and $\varphi_{\mathrm{NOT}}(2) = 13$.

Several bitwise operators are defined for the integral types in C++. There is a bitwise AND &, a bitwise OR |, and a bitwise XOR ^, as well as a bitwise NOT ~ that is usually referred to as *complement*. As the arithmetic operators, the binary bitwise operators (except for ~) have a corresponding assignment operator. The precedences and associativity of these operators are listed in Table 3.

Description	Operator	Arity	Prec.	Assoc.
bitwise complement	~	1	16	right
bitwise and	&	2	9	left
bitwise xor	^	2	8	left
bitwise or	|	2	7	left
and assignment	&=	2	4	right
xor assignment	^=	2	4	right
or assignment	|=	2	4	right

Table 3: *Precedence and associativity of bitwise operators.*

Note that the functionality of these operators is implementation defined, since the bitwise representations of integral type values are not specified by the C++ standard. We have only discussed the most frequent (and most likely) such representations in Section 2.2.8. You should therefore *only* use these operators when you *know* the representation. Even then, expressions involving the bitwise operators are implementation defined.

This is most obvious with the bitwise complement: even if we assume the standard binary representation of Section 2.2.8, the value of the expression ~0 depends on the number b of bits in the representation. This value therefore changes when you switch from a 32-bit machine to a 64-bit machine.

2.3.5 Goals

Dispositional. At this point, you should ...

1) know the basic terminology around Boolean functions and understand the concept of completeness;
2) know the type `bool`, its value range, and the conversions and operations involving `bool`;
3) understand the evaluation of expressions involving logical and relational operators, in particular the Boolean Evaluation Rule and the concept of short circuit evaluation.

Operational. In particular, you should be able to ...

(G1) prove or disprove basic statements about Boolean functions;

(G2) prove whether or not a given set of binary Boolean functions is complete;

(G3) evaluate a given expression involving arithmetic, logical, and relational operators;

(G4) read and understand a given simple program (see below), involving objects of arithmetic type (including `bool`) and arithmetic, logical, and relational operators.

The term *simple program* refers to a program that consists of a main function which in turn consists of a sequence of declaration and expression statements. Naturally, only the fundamental types and operations discussed in the preceding sections are used.

2.3.6 Exercises

Exercise 23 *For $n \in \mathbb{N}$, how many different Boolean functions $f : \mathcal{B}^n \to \mathcal{B}$ exist?* (G1)

Exercise 24 *Prove or disprove that for all $x, y, z \in \mathcal{B}$* (G1)

a) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$. (i.e., XOR is associative)

b) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$. (i.e., (AND, OR) is distributive)

c) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. (i.e., (OR, AND) is distributive)

d) $(x \uparrow y) \uparrow z = x \uparrow (y \uparrow z)$. (i.e., NAND is associative)

Exercise 25 *For $x_1, \ldots, x_n$, $n \in \mathbb{N}$, give a verbal description of $x_1 \oplus x_2 \oplus \ldots \oplus x_n$ in terms of the x_i, $1 \leq i \leq n$.* (G1)

Exercise 26 *Show that the following sets of functions are complete for the set of binary Boolean functions.* (G2)

a) $\{\text{AND}, \text{NOT}\}$

b) $\{\mathrm{OR}, \mathrm{NOT}\}$

c) $\{\mathrm{NAND}\}$

d) $\{\mathrm{NOR}\}$, where $\mathrm{NOR} := \mathrm{NOT} \circ \mathrm{OR}$.

e) $\{\mathrm{XOR}, \mathrm{AND}\}$

You may use the fact that $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NOT}\}$ is a complete set of binary Boolean functions.

Exercise 27 *Suppose* `a`, `b`, *and* `c` *are all variables of type* `int`. *Find values for* `a`, `b`, *and* `c` *for which the expressions* `a < b < c` *and* `a < b && b < c` *yield different results.* (G3)

Exercise 28 *Parenthesize the following expressions according to operator precedences and associativities.* (G3)

a) `x != 3 < 2 || y && -3 <= 4 - 2 * 3`

b) `z > 1 && ! x != 2 - 2 == 1 && y`

c) `3 * z > z || 1 / x != 0 && 3 + 4 >= 7`

Exercise 29 *Evaluate the expressions given in Exercise 28 step-by-step, assuming that* `x`, `y`, *and* `z` *are all of type* `int` *with* `x==0`, `y==1`, *and* `z==2`. (G3)

Exercise 30 *What can you say about the output of the following program? Characterize it depending on the input and explain your reasoning.* (G4)

```
#include <iostream>
int main()
{
  int a;
  std::cin >> a;
  std::cout << (a++ < 3) << ".\n";
  bool b = a * 3 > a + 4 && !(a >= 5);
  std::cout << (!b || ++a > 4) << ".\n";
  return 0;
}
```

Exercise 31 *Find the logical parentheses in lines 9 and 12 of the following program. What can you say about the output of the following program? Characterize it depending on the input and explain your reasoning.* (G4)

```
1 #include <iostream>
2
3 int main ()
4 {
```

```
    unsigned int a;
    std::cin >> a;

    unsigned int b = a;
    b /= 2 + b / 2;
    std::cout << b << "\n";

    bool c = a < 1 || b != 0 && 2 * a / (a - 1) > 2;
    std::cout << c << "\n";

    return 0;
}
```

2.3.7 Challenges

Exercise 32 *The* Reverse Polish Notation *(RPN) is a format of writing expressions without any parentheses. RPN became popular in the late nineteensixties when the company Hewlett-Packard started to use it as input format for expressions on their desktop and handheld calculators.*

In RPN, we first write the operands, and then the operator (that's what the Reverse *stands for). For example, the expression*

$$\mathrm{AND}(\mathrm{OR}(0, \mathrm{NOT}(\mathrm{AND}(0, 1))), 1)$$

can be written like this in RPN:

0 0 1 AND NOT OR 1 AND.

The latter sequence of operands and operators defines a specific evaluation sequence of the expression, see Section 2.2.3. To evaluate an expression in RPN, we go through the sequence from left to right; whenever we find an operand, we don't do anything, but when we find an operator (of arity n), we evaluate it for the n operands directly to the left of it and replace the involved $n + 1$ sequence elements by the result of the evaluation. Then we go to the next sequence element. In case of our example above, this proceeds as follows (currently processed operator in bold):

$$0\ \underbrace{0\ 1\ \textbf{AND}}_{0}\ \text{NOT OR 1 AND}$$

$$0\ \underbrace{0\ \textbf{NOT}}_{1}\ \text{OR 1 AND}$$

$$\underbrace{0\ 1\ \textbf{OR}}_{1}\ \text{1 AND}$$

$$\underbrace{1\ 1\ \textbf{AND}}_{1}$$

$$1$$

To see that this is indeed a way of evaluating the original expression

$$\mathrm{AND}(\mathrm{OR}(0, \mathrm{NOT}(\mathrm{AND}(0, 1))), 1),$$

you can for example make a bottom-up drawing of an expression tree (Section 2.2.2) that corresponds to the evaluation sequence in RPN. You will find that this tree is also valid for the original expression.

Here comes the actual exercise. Write programs and.C*,* or.C*, and* not.C *that receive as input a sequence* s *of boolean values in* $\{0, 1\}$ *("all operands to the left of the operator"). The output should be the sequence* s' *that we get by replacing the last* n *operands in* s *with the result of evaluating the respective operator for them. In case of* and.C *and* or.C*, we use* $n = 2$*, and for* not.C $n = 1$*. For example, on input* $(1, 1, 0)$*, program* and *should output the sequence* $(1, 0)$*, while* not *should yields* $(1, 1, 1)$*.*

In addition, write programs zero.C *and* one.C *that output the sequence* s' *obtained by appending a* 0 *or* 1 *to the input* s*. Finally, write a program* eval.C *(with no input) that outputs the empty sequence.*

The goal of all this is to evaluate boolean functions in RPN by simply calling the corresponding sequence of programs (preceded by a call to eval*), where the output of one program is used as input for the next one in the sequence. In Unix and Linux this can elegantly be done via a* pipe*. For example, to evaluate the example expression from above in RPN, we simply type the command*

```
./eval |./zero |./zero |./one |./and |./not |./or |./one |./and
```

This calls all listed programs in turn, where a separating pipe symbol | *has the effect that the output of the program to the left of it is used as the input for ("is piped into") the program to the right of it.*

Consequently, the whole aforementioned command should simply write 1 *to standard output, the result of the evaluation. Also test your programs with some other RPN sequences, in particular the "obvious" ones of the form*

```
./eval |./zero |./one |./or
```

(this one should output 1*) to make sure that they work as expected.*

Hint: *It is not necessary that your programs accept sequences* s *of arbitrary length as input. A maximum length of 32, for example, is sufficient for all practical purposes.*

2.4 Control statements

We are what we repeatedly do. Excellence, then, is not an act but a habit.

Will Durant in a summary of Aristotle's ideas,
The Story of Philosophy: The Lives and Opinions
of the World's Greatest Philosophers (1926)

This section introduces four concepts to control the execution of a program: selection, iteration, blocks, *and* jumps. *These concepts enable us to deviate from the default linear control flow which executes statement by statement from top to bottom. You will learn how these concepts are implemented in C++, and how to apply them to create interesting programs.*

The programs we have seen so far are all pretty simple. They consist of a sequence of statements that are executed one by one from the first to the last. Such a program is said to have a *linear control flow*. This type of control flow is quite restrictive, as each statement in the source code is executed at most once during the execution of the program. Suppose you want to implement an algorithm that performs 10,000 steps for some input. Then you would have to write a program with at least 10,000 lines of code. Obviously this is undesirable. Therefore, in order to implement non-trivial algorithms, more powerful mechanisms to control the flow of a program are needed.

2.4.1 Selection: if– and if-else statements

One particularly simple way to deviate from linear control flow is to select whether or not a particular statement is executed. In C++ this can be done via an `if` *statement*. The syntax is

```
if ( condition )
  statement
```

where *condition* is an expression or variable declaration of a type whose values can be converted to `bool`, and *statement* — as the name suggests — is a statement.[11] The semantics is the following: *condition* is evaluated; if and only if its value is *true*, *statement* is executed afterwards. In other words, an `if` statement splits the control flow into two branches. The value of *condition* selects which of these branches is executed. For example, the following lines of code

[11] In case you are missing a semicolon after *statement*: recall that this semicolon is part of the statement.

```
int a;
std::cin >> a;
if (a % 2 == 0) std::cout << "even";
```

read a number from standard input into the variable a and write "even" to standard output if and only if a is even.

Optionally, an if statement can be complemented by an else-*branch*. The syntax is

```
if ( condition )
  statement1
else
  statement2
```

and the semantics is as follows: *condition* is evaluated; if its value is *true*, *statement1* is executed afterwards; otherwise, *statement2* is executed afterwards. For example, the following lines of code

```
int a;
std::cin >> a;
if (a % 2 == 0)
  std::cout << "even";
else
  std::cout << "odd";
```

read a number from standard input into the variable a. Then if a is even, "even" is written to standard output; otherwise, "odd" is written to standard output.

When formatting an if statement, it is common to insert a line break before *statement1*, before else, and before *statement2*. Moreover, *statement1* and *statement2* are indented and else is aligned with if, as shown in the example above. If the whole statement fits on a single line then it can also be typeset as a single line.

Collectively, if- and if-else statements are known as *selection statements*.

2.4.2 Iteration: for statements

A much more powerful way of manipulating the control flow is provided by *iteration statements*. Iteration allows to execute a statement many times, possibly with different parameters each time. Iteration statements are also called *loops*, as they "loop through" a statement (potentially) several times. Selection and iteration statements are collectively referred to as *control statements*.

Consider the problem of computing the sum $S_n = \sum_{i=1}^{n} i$ of the first n natural numbers, for a given $n \in \mathbb{N}$. Program 7 reads in a variable n from standard input, defines another variable s to contain the result, computes the result and finally outputs it. In order to understand why the program sum_n.C indeed behaves as claimed, we have to explain the different parts of a for *statement*.

```
// Program: sum_n.C
// Compute the sum of the first n natural numbers.

#include <iostream>

int main()
{
  // input
  std::cout << "Compute the sum 1+...+n for n =? ";
  unsigned int n;
  std::cin >> n;

  // computation of sum_{i=1}^n i
  unsigned int s = 0;
  for (unsigned int i = 1; i <= n; ++i) s += i;

  // output
  std::cout << "1+...+" << n << " = " << s << ".\n";
  return 0;
}
```

Program 7: *progs/sum_n.C*

for statement. The `for` statement is a very compact form of an iteration statement, as it combines three statements or expressions into one. In most cases, the `for` statement serves as a "counting loop" as in Program 7. Its syntax is defined by

`for` (*init-statement condition*; *expression*)
  statement

where *init-statement* is an expression statement, a declaration statement, or the null statement, see Section 2.1.14. In all of these cases, *init-statement* ends with a semicolon such that there are always two semicolons in between the parentheses after a `for`. Usually *init-statement* defines and initializes a variable that is used to control and eventually end the iteration statement's execution. In `sum_n.C`, *init-statement* is a declaration statement that defines the variable `i`.

As in an `if` statement, *condition* is an expression or variable declaration whose type can be converted to `bool`. It defines how long the iteration goes on, namely as long as *condition* returns *true*. It is allowed that *condition* is empty in which case its value is interpreted as *true*. As the name suggests, *expression* is an arbitrary expression that may also be empty (in which case it has no effect). *statement* is an arbitrary statement. It is referred to as the *body* of the `for` statement.

Typically, *expression* has the effect of changing a value that appears in *condition*. Such an effect is said to "make progress towards termination". The goal is that *condition* is *false* after *expression* has been evaluated a finite number of times. In that sense, every evaluation of *expression* makes a step towards the end of the `for` statement. In `sum_n.C`, *expression* increments the variable `i` which is bounded from above in *condition*. In cases like this where the value of a single variable is accessed and changed by *condition* and *expression*, we call this variable the *control variable* of the `for` statement.

We are now ready to precisely define the semantics of a `for` statement. First, *init-statement* is executed once. Thereafter *condition* is evaluated. If it returns *true*, an *iteration* of the loop starts. If *condition* returns *false*, the `for` statement *terminates*, that is, its processing ends immediately.

Each single iteration of a `for` statement consists of first executing *statement* and then evaluating *expression*. After each iteration, *condition* is evaluated again. If it returns *true*, another iteration follows. If *condition* returns *false*, the `for` statement terminates. The execution order is therefore *init-statement*, *condition*, *statement*, *expression*, *condition*, *statement*, *expression*, ... until *condition* returns *false*.

Let's see this in action: Consider the `for` statement

```
for (unsigned int i = 1; i <= n; ++i) s += i;
```

in `sum_n.C` and suppose `n == 2`. First, the variable `i` is defined and initialized to 1. Then it is tested whether `i <= n`. As `1 <= 2` is *true*, the first iteration starts. The statement `s += i` is executed, setting `s` to 1, and thereafter `i` is incremented by one such that `i == 2`. One iteration is now complete. As a next step, the condition `i <= n` is evaluated again. As `2 <= 2` is *true*, another iteration follows. First `s += i` is executed, setting `s` to 3. Thereafter, `i` is incremented by one such that `i == 3`. The second iteration is now complete. The subsequent evaluation of `i <= n` entails `3 <= 2` which is *false*. Thus, no further iteration takes place and the processing of the `for` statement ends. The value of `s` is now 3, the sum of the first `n == 2` natural numbers.

Infinite loops. It is easily possible to create loops that do not terminate. For example, recall that both *condition* and *expression* may be empty. Moreover, both *init-statement* and *statement* can be the null statement. In this case we get the `for` statement

```
for (;;);
```

As the empty *condition* has value *true*, executing this statement runs through iteration after iteration without actually doing anything. Therefore, `for (;;)` may be read as "forever". In general, a statement which does not terminate is called an *infinite loop*.

Clearly, infinite loops are extremely undesirable and programmers try hard to avoid them. Nevertheless, sometimes such loops occur even in real life software. If you regularly use a computer, you have probably experienced this kind of phenomenon: a program "hangs".

You may ask: Why doesn't the compiler simply detect infinite loops and warns me about them just as it complains about syntax errors? Indeed, this would be a great thing to have and it would solve many problems in software development. The problem is that infinite loops are not always as easy to spot as in the above example. Loops can be pretty complicated, and possibly they loop infinitely when executed in certain program states only.

In fact, the situation is hopeless: It can be shown that the problem of detecting infinite loops (commonly referred to as the *halting problem*) cannot be solved by a computer, as we have and understand it today (see the Details). Therefore, some care is needed when designing loops. We have to check "by hand" that the iteration statement terminates for all possible program states that can occur.

Gauss. You may know or have realized that our program `sum_n.C` is actually a bad example. It is bad in the sense that it does not convincingly demonstrate the power of control statements.

In his primary school days, the German mathematician Carl Friedrich Gauss (1777–1855) was told to sum up the numbers $1, 2, 3, \ldots, 100$. The teacher had planned to keep his students busy for a while, but Gauss came up with the correct result 5050 very quickly. He had imagined writing down the numbers in increasing order, and one line below once again in decreasing order. Clearly, the two numbers in each column sum up to 101; hence, the overall sum is $100 \cdot 101 = 10100$, half of which is the number that was asked for.

$$\begin{array}{rrrcrrr} 1 & 2 & 3 & \ldots & 98 & 99 & 100 \\ 100 & 99 & 98 & \ldots & 3 & 2 & 1 \\ \hline 101 & 101 & 101 & \ldots & 101 & 101 & 101 \end{array}$$

In this way, Gauss discovered the formula

$$\sum_{i=1}^{n} i = n(n+1)/2,$$

for any $n \in \mathbb{N}$. The `for` statement in `sum_n.C` can therefore be replaced by the much more elegant and efficient statement [12]

```
s = n * (n + 1) / 2;
```

We next get to a real application of selection and iteration statements.

Prime numbers. In the introductory Section 1.1, we have talked a lot about prime numbers. How would a program look like that tests whether or not a given number is prime? According to the usual definition, a number $n \in \mathbb{N}, n \geq 2$ is prime if and only if it is not divisible by any number $d \in \{2, \ldots, n-1\}$. The strategy for our program is therefore clear: Write a loop that runs through all these numbers, and test each of them for being a divisor of n. If a divisor is found, we can stop and output a factorization of n into two

[12] Note that in this statement, the integer division coincides with the real division, since for all n, the product $n(n+1)$ is even.

numbers, proving that n is not prime. Otherwise, we output that n is prime. Program 8 implements this strategy in C++, using one `for` statement, and one `if` statement. Remarkably, the `for` statement has an empty body, since we have put the divisibility test into the *condition*. The important observation is that the *condition* `n % d != 0` definitely returns *false* for `d == n`, so that the loop is guaranteed to terminate; if (and only if) *condition* returns *false* earlier, we have found a divisor of n in the range $\{2, \ldots, n-1\}$.

```
// Program: prime.C
// Test if a given natural number is prime.

#include <iostream>

int main ()
{
  // Input
  unsigned int n;
  std::cout << "Test if n>1 is prime for n =? ";
  std::cin >> n;

  // Computation: test possible divisors d
  unsigned int d;
  for (d = 2; n % d != 0; ++d);

  // Output
  if (d < n)
    // d is a divisor of n in {2,...,n-1}
    std::cout << n << " = " << d << " * " << n / d << ".\n";
  else
    // no proper divisor found
    std::cout << n << " is prime.\n";

  return 0;
}
```

Program 8: *progs/prime.C*

2.4.3 Blocks and scope

In C++ it is possible to group a sequence of one or more statements into one single statement that is then called a *compound statement*, or simply a *block*. This mechanism does not manipulate the control flow directly. Blocks allow to structure a program by grouping statements that logically belong together. In particular, they are a tool to design powerful and at the same time readable control statements.

Syntactically, a block is simply a sequence of zero or more statements that are enclosed in curly braces.

{ *statement1 statement2 ... statementN* }

Each of the statements may in particular be a block, so it is possible to have nested blocks. The simplest block is the empty block {}.

You have already seen blocks. Each program contains a special block, the so-called *function body* of the `main` function. This block encloses the sequence of statements that is executed when the `main` function is called by the operating system.

Using blocks, one can create selection and iteration statements whose body contains a sequence of two or more statements. For example, suppose that for testing purposes we would like to write out all partial sums during the computation in `sum_n.C`:

```
for (unsigned int i = 1; i <= n; ++i) {
  s += i;
  std::cerr << i << "-th partial sum is " << s << "\n";
}
```

Here two statements are executed in each iteration of the loop. First, the next summand is added to `s`, then the current value of `s` is written to standard error output.

Blocks should in general be formatted as shown above. That is, a line break appears after the opening and before the closing brace, and all lines in between are indented one level. Only if the block consists of just one single statement and it all fits on one line, the block can be formatted as one single line.

The type of test output we have created in the previous example is called *debugging output*. A *bug* is a commonly used term to denote a programming error, hence "debugging" is the process of finding and eliminating such errors. It is good practice to write debugging output to standard error output since it can then more easily be separated from the "real" program output that usually goes to standard output.

Visibility. Blocks do not only structure a program visually but they also provide a logical boundary around declarations (of variables, for example). Any declaration that appears inside a block is called *local* to that block. A local declaration extends only until the end of the block in which it appears. A name that is introduced by a local declaration is not "visible" outside of the block where it is declared. For example, in

```
int main()
{
  {
    int i = 2;
  }
  std::cout << i; // error, undeclared identifier
  return 0;
}
```

the variable `i` declared inside the block in line 3–5 is not visible in the output statement in line 6. Thus, if you confront the compiler with this code, it issues an error message.

Control statements and blocks. Control statements act like blocks themselves. Therefore any declaration appearing in a control statement is local to that control statement. In particular, this applies to a variable defined in the *init-statement* of a `for` statement. For example, in

```
1 int main()
2 {
3   for (unsigned int i = 0; i < 10; ++i) s += i;
4   std::cout << i; // error, undeclared identifier
5   return 0;
6 }
```

the expression `i` in line 4 does *not* refer to the variable `i` defined in line 3.

Declarative region. After having seen these first examples, we will now introduce the precise terminology that allows us to deduce which names can be used where in the program. Each declaration has an associated *declarative region*. This region is the part of the program in which the declaration appears. Such a region can be a block, a function definition, or a control statement. In all these cases the declaration is said to have *local scope*. A declaration can also have *namespace scope*, if it appears inside a namespace, see Section 2.1.3. Finally, a declaration that is outside of any particular other structure has *global scope*.

Scope. A name introduced by a declaration D is *valid* or *visible* in a part of its declaration's declarative region, called the *scope* of the declaration. Within the scope of D, the name introduced by D may be used and actually refers to the declaration D. In most cases, the scope of a declaration is equal to its *potential scope*.

The *potential scope* of a declaration starts at the point where the declaration appears. For the name to be declared this is called its *point of declaration*. The potential scope extends until the end of the declarative region.

To get the scope of a declaration, we start from its potential scope but we possibly have to remove some parts of it. This happens when the potential scope contains one or more declarations of the *same* name. As an example, consider Program 9.

```
#include <iostream>

int main()
{
  int i = 2;
  for (int i = 0; i < 5; ++i)
    std::cout << i;   // outputs 0, 1, 2, 3, 4
```

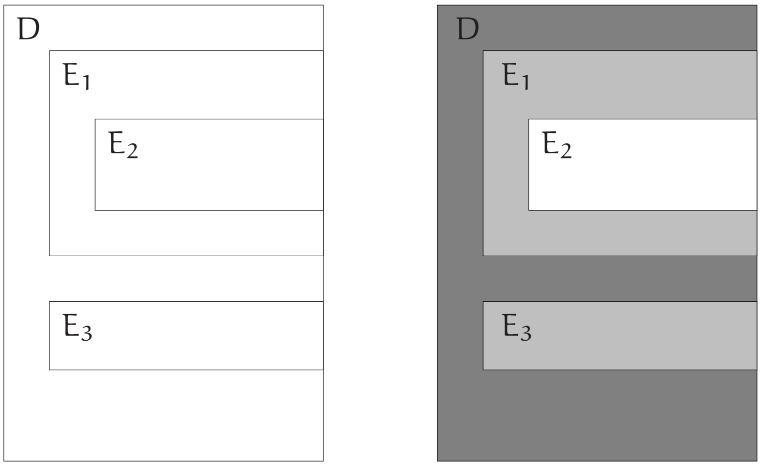


Figure 5: *Potential scopes of declarations* D, E_1, E_2, E_3 *of the same name, drawn as rectangles with the corresponding declaration in the upper left corner (left); on the right, we see the resulting scopes of* D *(dark gray),* E_1, E_3 *(light gray) and* E_2 *(white).*

```
 8   std::cout << i;      // outputs 2
 9   return 0;
10 }
```

Program 9: *progs/scope.C*

The i in line 7 refers to the declaration from line 6, whereas the i in line 8 refers to the declaration from line 5. Therefore, the program outputs first 0, 1, 2, 3, 4, and then 2. In some sense, the declaration in line 6 temporarily hides the previous declaration of i from line 5. This phenomenon is called *name hiding*. But when the declarative region of the second declaration ends in line 7, the second declaration "becomes invisible" (we say: "it runs out of scope") and the first declaration takes over again. In particular, since the name i in line 8 refers to the variable defined in line 5, we get the output 2 in line 8.

It is good practice to avoid name hiding since this unnecessarily obfuscates the program. On the other hand, name hiding allows us (like in Program 9) to use our favorite identifier i as the name of the control variable in a for statement, without having to check whether there is some other name i somewhere else in the program. This is an acceptable and even useful application of name hiding.

Now we can get to the formal definition of scope in the general case (possible presence of multiple declarations of the same name). The *scope* of a declaration D is obtained from its potential scope as follows: For each declaration E in the potential scope of D such that both D and E declare the same name, the potential scope of E is removed from the scope of D. Figure 5 gives a symbolic picture of the situation.

In Program 9, the declarative region of the declaration in line 5 is line 4–10 (a block), its potential scope is line 5–10, and its scope is line 5 plus line 8–10. For the declaration in line 6, declarative region (a control statement), potential scope and scope are line 6–7.

Breaking down the scopes into lines is in general not possible, of course, since line breaks may (or may not) appear almost anywhere. If we want to talk about scope on a line-by-line basis, we have to format the program accordingly.

Storage duration. Related to the scope of a variable is its *storage duration*. This term denotes the time in which the address of the variable is valid, that is, some memory location is assigned to it.

For a variable with local scope, the storage duration is usually the time in which the program's control is in the variable's potential scope. During program execution, this means that whenever the variable declaration is reached, some memory location is assigned and the address becomes valid. And whenever the execution gets to the end of the declarative region, the associated memory is freed and the variable's address becomes invalid.[13] We therefore get a "fresh instance" of the variable everytime its declaration is executed.

This behavior is called *automatic storage duration*. For example, in

```
for (unsigned int i = 0; i < 10; ++i) {
  int k = 2;
  // do something with k
}
```

the address of the variable `k` may change in each iteration of the loop. Also the initialization to 2 takes place in each iteration.

As a more concrete example, consider the following code fragment.

```
1 int i = 5;
2 for (int j = 0; j < 5; ++j) {
3   std::cout << ++i; // outputs 6, 7, 8, 9, 10
4   int k = 2;
5   std::cout << --k; // outputs 1, 1, 1, 1, 1
6 }
```

Since line 3 belongs to the scope of the declaration in line 1, the effect of line 3 is to increment the variable defined in line 1 in every iteration of the `for` statement. Line 5, on the other hand, belongs to the scope of the declaration in line 4; the effect of line 5 is therefore to decrement the "fresh" variable `k` in every iteration, and this always results in value 1.

In contrast, a variable that is defined in namespace scope or global scope has *static storage duration*. This means that its address is determined at the beginning of the program's execution, and it does not change (hence "static") nor become invalid until the execution of the program ends. The variables named by `std::cin` and `std::cout`,

[13]Note that the address does not necessarily remain the same throughout the program's execution.

for instance, have static storage duration. Variables with static storage duration are also referred to as *static variables*.

2.4.4 Iteration: while statements

So far, we have seen one iteration statement, the `for` statement. The `while` *statement* is a simplified `for` statement, where both *init-statement* and *expression* are omitted. Its syntax is

`while` (*condition*)
  statement

where *condition* and *statement* are as in a `for` statement. As before, *statement* is referred to as the body of the `while` statement. Semantically, a `while` statement is equivalent to the corresponding `for` statement

`for` (; *condition* ;)
  statement

The execution order is therefore *condition*, *statement*, *condition*,… until *condition* returns *false*.

Since `while` statements are so easy to rewrite as `for` statements, why do we need them? The main reason is readability. As its name suggests, a `for` statement is typically perceived as a counting loop in which the increment (or decrement) of a single variable is responsible for the progress towards termination. In this case, the progress is most conveniently made in the `for` statement's *expression*. But the situation can be more complex: the progress may depend on the values of several variables, or on some condition that we check in the loop's body. In some of these cases, a `while` statement is preferable. The next section describes an example.

The Collatz problem. Given a natural number $n \in \mathbb{N}$, we consider the *Collatz sequence* $n_0, n_1, n_2, \ldots$ with $n_0 = n$ and

$$n_i = \begin{cases} n_{i-1}/2, & \text{if } n_{i-1} \text{ is even} \\ 3n_{i-1} + 1, & \text{if } n_{i-1} \text{ is odd} \end{cases} \qquad i \geq 1.$$

For example, if $n = 5$, we get the sequence $5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots$. Since the sequence gets repetitive as soon as 1 appears, we may stop at this point. Program 10 reads in a number n and outputs the elements of the sequence $(n_i)_{i \geq 1}$ until the number 1 appears.

```
// Program: collatz.C
// Compute the Collatz sequence of a number n.

#include <iostream>
```

```
int main()
{
  // Input
  std::cout << "Compute the Collatz sequence for n =? ";
  unsigned int n;
  std::cin >> n;

  // Iteration
  while (n > 1) {
    if (n % 2 == 0)
      n = n / 2;
    else
      n = 3 * n + 1;
    std::cout << n << " ";
  }
  std::cout << "\n";
  return 0;
}
```

Program 10: *progs/collatz.C*

The loop can of course be written as a `for` statement with empty *init-statement* and *expression*, but the resulting variant of the program is less readable since it tries to advertise the rather complicated iteration as a simple counting loop. As a rule of thumb, if there is a simple *expression* that captures the loop's progress, use a `for` statement. Otherwise, consider formulating your loop as a `while` statement.

Talking about progress: is it clear that the number 1 always appears? If not, the program `collatz.C` contains an infinite loop for certain values of `n`. If you play with the program, you will observe that 1 indeed appears for all numbers you try, although this may take a while. You will find, for example, that the Collatz sequence for $n = 27$ is

$$27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

It is generally believed that 1 eventually comes up for all values of n, but mathematicians have not yet been able to produce a proof of this conjecture. As innocent as it looks, this problem seems to be a very hard mathematical nut to crack (see also the Details section), but you are certainly invited to give it a try!

2.4.5 Iteration: do statements

Do *statements* are similar to while statements, except that the condition is evaluated *after* every iteration of the loop instead of *before* every iteration. Therefore, in contrast to for– and while statements, the body of a do statement is executed at least once.
The syntax of a do statement is as follows.

```
do
  statement
while ( expression );
```

where *expression* is of a type whose values can be converted to bool.

The semantics is defined as follows. An iteration of the loop consists of first executing *statement* and then evaluating *expression*. If *expression* returns *true* then another iteration follows. Otherwise, the do statement terminates. The execution order is therefore *statement*, *expression*, *statement*, *expression*, ... until *expression* returns *false*.

Alternatively, the semantics could be defined in terms of the following equivalent for statement.

```
for ( bool firsttime = true; firsttime || expression; firsttime = false )
  statement
```

This behaves like our "simulation" of the while statement, except that in the first iteration, *expression* is not evaluated (due to short circuit evaluation, see Section 2.3.3), and *statement* is executed unconditionally.

Consider a simple calculator-type application in which the user enters a sequence of numbers, and after each number the program outputs the sum of the numbers entered so far. By entering 0, the user indicates that the program should stop. This is most naturally written using a do statement, since the termination condition can only be checked *after* the next number has been entered.

```
int a;      // next input value
int s = 0; // sum of values so far
do {
  std::cout << "next number =? ";
  std::cin >> a;
  s += a;
  std::cout << "sum = " << s << "\n";
} while (a != 0);
```

In this case, it is *not* possible to declare a where we would usually do it, namely immediately before the input statement. The reason is that a would then be local to the body of the do statement and would not be visible in the do statement's *expression* a != 0.

2.4.6 Jump statements

At this point, we would like to extend our arsenal of control statements with a special type of statements that are referred to as *jump statements*. These statements are not necessary in the sense that they would allow you to do something which is not possible otherwise. Instead, just like `while`– and `do` statements (which are also unnecessary in that sense), jump statements provide additional flexibility in designing iteration statements. You should use this flexibility wherever it allows you to improve your code. However, be also warned that jump statements should be used with care since they tend to complicate the control flow. The complication of the control flow has to be balanced by a significant gain in one of the other categories. Therefore, think carefully before introducing a jump statement!

When a jump statement is executed, the program flow unconditionally "jumps" to a certain point. There are two different jump statements that we want to discuss here.

The first jump statement is called a `break` *statement*; its syntax is rather simple.

```
break;
```

When a `break` statement is executed within an iteration statement, [14] the smallest enclosing iteration statement terminates immediately. The execution continues at the statement after the iteration statement (if any). For example,

```
for (;;) break;
```

is not an infinite loop but rather a complicated way of writing a null statement. Here is a more useful appearance of `break`. In our calculator example from Page 88, it would be more elegant to suppress the irrelevant addition of 0 in the last iteration. This can be done with the following loop.

```
for (;;) {
  std::cout <<  "next number =? ";
  std::cin >> a;
  if (a == 0) break;
  s += a;
  std::cout << "sum = " << s << "\n";
}
```

Here, we see the typical usage of `break`, namely the termination of a loop "somewhere in the middle". Note that we could equivalently write

```
do {
  std::cout <<  "next number =? ";
  std::cin >> a;
  if (a == 0) break;
  s += a;
```

[14] otherwise, it can only occur in a `switch` statement, see the Details.

```
  std::cout << "sum = " << s << "\n";
} while (true);
```

In this case for is preferable, though, since it nicely reads as "forever". Of course, the same functionality is possible without break, but the resulting code requires an additional block and evaluates a != 0 twice.

```
do {
  std::cout <<  "next number =? ";
  std::cin >> a;
  if (a != 0) {
    s += a;
    std::cout << "sum = " << s << "\n";
  }
} while (a != 0);
```

The second jump statement is called a continue *statement*; again the syntax is simple.

```
continue;
```

When a continue statement is executed, the remainder of the smallest enclosing iteration statement's body is skipped, and execution continues at the end of the body. The iteration statement itself is *not* terminated.

If the surrounding iteration statement is a while– or do statement, the execution therefore continues by evaluating its *condition*. If the surrounding iteration statement is a for statement, the execution continues by evaluating its *expression* and then its *condition*. Like the break statement, the continue statement can therefore be used to manipulate the control flow "in the middle" of a loop.

In our calculator example, the following variant of the loop ignores negative input. Again, it would be possible to do this without continue, at the expense of another nested block.

```
for (;;) {
  std::cout <<  "next number =? ";
  std::cin >> a;
  if (a < 0) continue;
  if (a == 0) break;
  s += a;
  std::cout << "sum = " << s << "\n";
}
```

2.4.7 Equivalence of iteration statements

In terms of pure functionality, the while– and do statements are redundant, as both of them can equivalently be expressed using a for statement. This may create the impression that for statements have more expressive power than while– and do statements.

In this section we show that this is not the case: all three iteration statements are functionally equivalent. More precisely, we show how to use

- `do` statements to express `while` statements, and
- `while` statements to express `for` statements.

If we denote "A can be used to express B" by A ⇒ B, we therefore have

`do` statement ⇒ `while` statement ⇒ `for` statement ⇒ `do` statement,

where we know the last implication from the previous section. Together, this clearly "proves" the claimed equivalence.

Note that we put the word *proves* in quotes, as our reasoning cannot be considered a formal proof. In order to really prove a statement like this, we first of all would have to be more formal in defining the semantics of statements. Semantics of programming languages is a subject of its own, and the formal treatment of semantics is way beyond what we can do here. In other words: The following is as much of a "proof" as you will get here, but it is sufficient to understand the relations between the three iteration statements.

do statement ⇒ while statement. Consider the `while` statement

```
while ( condition )
  statement
```

Your first idea how to simulate this using a `do` statement might look like this:

```
if ( condition )
  do
    statement
  while ( condition );
```

Indeed, this induces the execution order *condition*, *statement*, *condition*,... until *condition* returns *false* and the statement terminates. But there is a simple technical problem: if *condition* is a variable declaration, we can't use it as the *expression* in the `do` statement. Here is a reformulation that works. [15]

```
do
  if ( condition )
    statement
  else
    break;
while ( true );
```

This induces exactly the `while` statement's execution order *condition*, *statement*, *condition*,... until *condition* returns *false* and the loop is terminated using `break`.

[15] We are not saying that this should be done in practice. On the contrary, this should *never* be done in practice. This section is about *conceptual* equivalence, not about practical equivalence.

while statement ⇒ for statement. Simulating the `for` statement

```
for ( init-statement condition; expression )
  statement
```

by a `while` statement seems easy:

```
{
  init-statement
  while ( condition ) {
    statement
    expression;
  }
}
```

Indeed, this will work, *unless statement* contains a `continue`. In the `for` statement, execution would then proceed with the evaluation of *expression*, but in the simulating `while` statement, *expression* is skipped, and *condition* comes next. This reformulation is therefore wrong. Here is a version that works:

```
{
  init-statement
  while ( condition ) {
    bool b = false;
    while ( b = !b )
      statement
    if ( b ) break;
    expression;
  }
}
```

This looks somewhat more complicated, so let us explain what is going on.

We may suppose that the identifier `b` does not appear in the given `for` statement (otherwise we choose a different name). Note that the whole statement forms a separate block, as does a `for` statement. A potential declaration in *init-statement* as well as the scope of `b` is thus limited to this block.

Consider an execution of the outer `while` statement. First, *condition* is evaluated, and if it returns *false* the statement terminates. Otherwise, the variable `b` is set to *true* in the inner `while` statement's condition, meaning that *statement* is executed next. [16] If *statement* does not contain a `break`, the inner loop evaluates its condition for the second time. In doing so, `b` is set to *false*, and the condition returns *false*. Therefore, the inner loop terminates. Since `b` is now *false*, *expression* is evaluated next, followed by *condition*. This induces the `for` statement's execution order *condition*, *statement*, *expression*, *condition*,... until *condition* returns *false* and the outer loop terminates.

[16] Recall that the assignment operator returns the new value of its left operand.

In the case where *statement* contains a `break`, the inner loop terminates immediately, and `b` remains *true*. In this case, we also terminate the outer loop that represents our original `for` statement.

In retrospect, we should now check that jump statements cause no harm in our previous reformulation of the `while` statement in terms of the `do` statement. We leave this as an exercise.

2.4.8 Choosing the "right" iteration statements

We have seen that from a functional point of view, the `for` statement, the `while` statement and the `do` statement are equivalent. Moreover, the `break` and `continue` statements are redundant. Still, C++ offers all of these statements, and this gives you the freedom (but also the burden) of choosing the appropriate control statements for your particular program.

Writing programs is a dynamic process. Even though the program may do what you want at some point, the requirements change, and you will keep changing the program in the future. Even if there is currently no need to change the functionality of the program, you may want to replace a complicated iteration statement by an equivalent simpler formulation. The general theme here is *refactoring*: the process of rewriting a program to improve its readability or structure, while keeping its functionality unchanged.

Here is a simple guideline for writing "good" loops. Choose the loop that leads to the most *readable* and *concise* formulation. This means

- few statements,
- few lines of code,
- simple control flow, and
- simple expressions.

Almost never there is *the* one and only best formulation; however, there are always arguably bad choices which you should try to avoid. Usually, there are some tradeoffs, like fewer lines of code versus more complicated expressions, and there is also some amount of personal taste involved. You should experience and find out what suits you best.

Let us look at some examples to show what we mean. Suppose that you want to output the odd numbers between 0 and 100. Having just learned about the `continue` statement, you may write the following loop.

```
for (unsigned int i = 0; i < 100; ++i) {
  if (i % 2 == 0) continue;
  std::cout << i << "\n";
}
```

This is perfectly correct, but the following version is preferable since it has **fewer statements** and **fewer lines of code**.

```
for (unsigned int i = 0; i < 100; ++i)
  if (i % 2 != 0) std::cout << i << "\n";
```

This variant still contains nested control statements; but you can get rid of the `if` statement and obtain code with **simpler control flow**.

```
for (unsigned int i = 1; i < 100; i += 2)
  std::cout << i << "\n";
```

The same output can be produced with a `while` statement and equally simple control flow.

```
int i = -1;
while ((i += 2) < 100)
  std::cout << i << "\n";
```

But here, the condition is more complicated, since it combines assignment and comparison operators. Such expressions are comparatively difficult to understand due to the effect of the assignment operation. Also, the initialization of `i` to -1 is counter-intuitive, given that we deal with natural numbers.

You can solve the latter problem and at the same time get **simpler expressions** by writing

```
unsigned int i = 1;
while (i < 100) {
  std::cout << i << "\n";
  i += 2;
}
```

The price to pay is that you get less concise code; there are now five lines instead of the two lines that the `for` statement needs. It seems that for the simple problem of writing out odd numbers, a `for` statement with *expression* `i += 2` is the loop of choice.

2.4.9 Details

Nested if-else statements. Consider the statement

```
if(true) if (false); else std::cout << "Where do I belong?";
```

It is not a priori clear what its effect is: if the `else` branch belongs to the outer `if`, there will be no output (since the condition has value *true*), but if the `else` branch belongs to the inner `if`, we get the output `Where do I belong?`

The intuitive rule is that the `else` branch belongs to the `if` immediately preceding it, in our case to the inner `if`. Therefore, the output is `Where do I belong?`, and we should actually format the statement like this:

```
 if(true)
    if (false)
```

```
      ; // null statement
    else
      std::cout << "Where do I belong?";
```

Whenever you are unsure about rules like this, you can make the structure clear through explicit blocks:

```
  if(true) {
    if (false) {
      ; // null statement
    }
    else {
      std::cout << "Where do I belong?";
    }
  }
```

The switch statement. Besides `if...else` there exists a second selection statement in C++: the `switch` *statement*. It is useful to select between many alternative statements, using the following syntax.

`switch` (*condition*)
  statement

The value of *condition* must be convertible to an integral type. This is in contrast to the other control statements where *condition* has to be convertible to `bool`.

statement is usually a block that contains several *labels* of the form

`case` *literal*:

where *literal* is a literal of integral type. For no two labels shall these literals have the same value. There can also be a label `default:`.

The semantics of a `switch` statement is the following. *condition* is evaluated and the result is compared to each of the literals which appear in a label in *statement*. If for any of them the values agree, the execution continues at the statement immediately following the label. If there is no agreement but a `default:` label, the execution continues at the statement immediately following the `default:` label. Otherwise, *statement* is ignored and the execution continues after the `switch` statement.

Note that `switch` only selects an entry point for the processing of *statement*, it does not exit when the execution reaches another label. If one wants to separate the different alternatives, one has to use `break` (and this is the only legal use of `break` outside of an iteration statement). Consider for example the following piece of code, and let us suppose that `x` is a variable of type `int`.

```
switch (x) {
  case 0: std::cout << "0";
  case 1: std::cout << "1"; break;
```

```
  default: std::cout << "whatever";
}
```

For x==0 the output is 01; for x==1 the output is 1; otherwise we get the output whatever.

The switch statement is powerful in the sense that it allows the different alternatives to share code. However, this power also makes switch statements hard to read and error prone. A frequent problem is that one forgets to put a break where there should be one. Therefore, we mention switch here for completeness only. Whenever there are only a few alternatives to be distinguished, play it safe and use if...else rather than switch.

The Halting Problem, Decidability, and Computability. The halting problem is one of the fundamental problems in the theory of computation. Informally speaking, the problem is to decide (using an algorithm) whether a given program halts (terminates) when executed on a given input (program state). The term "program" may refer to a C++-program, but also to a program in any other common programming language.

To attack the problem formally, the British mathematician Alan Turing (1912-1954) defined in a seminal paper a "minimal" programming language; a program in this language is known as a *Turing machine*.

Turing proved that the halting problem is undecidable for Turing machines, but the same arguments can also be used to prove the same statement for C++ programs.

What does "undecidable" mean? We have seen a simple loop for which it was painfully evident that it is an infinite loop, haven't we? Yes, indeed one *can* decide the halting problem for many concrete programs. Undecidable means that (in a particular model of computation) there cannot be an algorithm that decides the halting problem for *all possible* programs.

Despite their simplicity, Turing machines are a widely accepted model of computation; in fact, just like machine language, Turing machines can do everything that C++ programs can do, except that they usually need a huge number of very primitive operations for that.

At the same time as Turing, the American mathematician Alonzo Church (1903–1995) developed a computational model called λ-calculus. As it turned out, his model is equivalent to Turing machines in terms of computational power. The Church-Turing thesis states that "every function that is naturally regarded as computable can be computed by a Turing machine". As there is no rigorous definition of what is "naturally regarded as computable", this statement is not a theorem but a hypothesis that cannot be proven mathematically. As of today, the hypothesis has not been disproved. In theoretical computer science the term *computable* used without further qualification is a synonym for "computable by a Turing machine" (equivalently, a C++ program).

Point of declaration. Our approach of defining potential scope and scope line by line is a simplification, even if the code is suitably formatted and we only have one declaration per line. The truth is that the point of declaration of i in

```
int i = 5;
```

is in the *middle* of the declaration, after the name `i` has appeared. The potential scope therefore does not include the full line, but only the part starting from `=`. This explains what happens in the following code fragment, but fortunately this is consistent with our line-by-line approach. In

```
1  int i = 5;
2  {
3    int i = i;
4  }
```

the name `i` after the `=` in line 3 refers to the declaration in line 3. Consequently, `i` is initialized with itself in this line, meaning that its value will be undefined, and not 5.

In other situations it may happen, though, that the appearance of a name in the declaration of the same name refers to a *previous* declaration of this name. For now, we can easily avoid such subtleties by the following rule: any declaration should contain the name to be declared only once.

The Collatz problem and the ?-operator. The Collatz sequence goes back to the German mathematician Lothar Collatz (1910–1990) who studied it in the 1930's. Several prizes have been offered to anyone who proves or disproves the conjecture that the number 1 appears in the Collatz sequence of every number $n \geq 1$. The famous Hungarian mathematician Paul Erdős (1913–1996) offered $500, which is much by his standards (he used to offer much lower amounts for very difficult problems). Erdős said that "Mathematics is not yet ready for such problems". Indeed, the conjecture is still unsolved.

We have presented the computation of the Collatz sequence as an application of the `while` statement, pointing out that the conditional change of `n` is too complicated to put it into a `for` statement's *expression*. Well, that's not exactly true: the designers of C, the precursor to C++, had a weakness for very compact code and came up with the *conditional operator* that allows us to simulate `if` statements by expressions. The syntax of this *ternary* operator (arity 3) is

condition ? *expression1* : *expression2*

Here, *condition* is an expression of a type whose values can be converted to `bool`, and *expression1* and *expression2* are expressions. The semantics is as follows. First, *condition* is evaluated. If it returns *true*, *expression1* is evaluated, and its value is returned as the value of the composite expression. Otherwise (if *condition* returns *false*), *expression2* is evaluated, and its value is returned. The token `?` is a sequence point (see Section 2.2.10), meaning that all effects of *condition* are processed before either *expression1* or *expression2* are evaluated.

Using the conditional operator, the loop of Program 10 could quite compactly be written as follows.

```
for ( ; n > 1; std::cout << (n % 2 == 0 ? n=n/2 : n=3*n+1) << " ");
```

We leave it up to you to decide whether you like this variant better.

Static variables. The discussion about storage duration above does not tell the whole story: it is also possible to define variables with local scope that have static storage duration.

This is done by prepending the keyword `static` to the variable declaration. For example, in

```
for (int i = 0; i < 5; ++i) {
  static int k = i;
  k += i;
  std::cout << k << "\n";
}
```

the address of `k` remains the same during all iterations, and `k` is initialized to `i` *once only*, in the first iteration. The above piece of code will therefore output the sequence of values $0, 1, 3, 6, 10$ (remember Gauss). Without the `static` keyword, the result would simply be the sequence of even numbers $0, 2, 4, 6, 8$.

Static variables have been quite useful in C, for example to count how often a specific piece of code is executed; in C++, they are less important.

For variables of fundamental type the initial value may be undefined, as in the definition `int x;`. However, the value is undefined only if `x` has automatic storage duration. In contrast, variables with static storage duration are always *zero-initialized*, that is, filled with a "zero" of the appropriate type.

Jump statements. There are two more jump statements in C++ that we haven't discussed in this section. One of them is the `return` statement that you already know (Section 2.1.14): it may occur only in a function, and its execution lets the program flow jump to the end of the corresponding function body. The other jump statement is the `goto` statement, but since this one is rarely needed (and somewhat difficult to use), we omit it.

2.4.10 Goals

Dispositional. At this point, you should ...

1) know the syntax and semantics of `if...else`–, `for`–, `while`–, and `do` statements;
2) understand the concepts block, selection, iteration, declarative region, scope, and storage duration;
3) understand the concept of an infinite loop and be aware of the difficulty of detecting such loops;
4) understand the conceptual equivalence of for–, while–, and do statements;
5) know the syntax and semantics of `continue`– and `break` statements;
6) know at least four criteria to judge the code quality of iteration statements.

Operational. In particular, you should be able to ...

(G1) check a given simple program (as defined below) for syntactical correctness and point out possible errors;

(G2) read and understand a given simple program and explain what happens during its execution;

(G3) find (potential) infinite loops in a given simple program;

(G4) find the matching declaration for a given identifier;

(G5) determine declarative region and scope of a given declaration;

(G6) reformulate a given `for`–, `while`–, or `do` statement equivalently using any of the other two statements;

(G7) compare the code quality of two given iteration statements and pick the one that is preferable (if any);

(G8) design simple programs for given tasks.

The term *simple program* refers to a program that consists of a main function in which up to four (possibly nested) iteration statements appear, plus some selection statements. Naturally, only the fundamental types and operations discussed in the preceding sections are used.

2.4.11 Exercises

Exercise 33 *Correct all syntax errors in the program below. What does the resulting program output for the following inputs?*
(a) -4 *(b)* 0 *(c)* 1 *(d)* 3 (G1)(G2)

```
#include <iostraem>
int main()
{
  unsinged int x = +1;
  { std::cin >> x; }
  for (int y = 0u; y < x) {
    std:cout << ++y;
  return 0;
}
```

Exercise 34 *What is the problem with the code below? Fix it and explain what the resulting code computes.* (G2)(G3)

```
unsigned int s = 0;
do {
  int i = 1;
  if (i % 2 == 1) s *= i;
} while (++i < 10);
```

Exercise 35 *For each variable declaration in the following program give its declarative region and its scope in the form "line x–y". What is the output of the program?* (G2)(G5)

```
1  #include <iostream>
2  int main()
3  {
4    int s = 0;
5    {
6      int i = 0;
7      while (i < 4)
8      {
9        ++i;
10       int f = i + 1;
11       s += f;
12       int s = 3;
13       i += s;
14     }
15     unsigned int t = 2;
16     std::cout << s + t << "\n";
17   }
18   int k = 1;
19   return 0;
20 }
```

Exercise 36 *Consider the program given below for each of the listed input numbers. Determine the values of* x, s, *and* i *at begin of the first five iterations of the for-loop, before the condition is evaluated. What does the program output for these inputs? (a)* -1 *(b)* 1 *(c)* 2 *(d)* 3 (G2)(G3)

```
#include <iostream>
int main()
{
  int x;
  std::cin >> x;
  int s = 0;
  for (int i = 0; i < x; ++i) {
    s += i;
    x += s / 2;
  }
  std::cout << s << "\n";
  return 0;
}
```

Exercise 37 *Find at least four problems in the code given below.* (G3)(G4)(G5)

```
#include <iostream>
int main()
{
  { unsigned int x; }
  std::cin << x;
  unsigned int y = x;
  for (unsigned int s = 0; y >= 0; --y)
    s += y;
  std::cout << "s=" << s << "\n";
  return 0;
}
```

Exercise 38 *For which input numbers is the output of the program given below well defined? List those input/output pairs and argue why your list is complete.* (G3)(G4)(G5)

```
#include <iostream>
int main()
{
  unsigned int x;
  std::cin >> x;
  int s = 0;
  for (unsigned int y = 1 + x; y > 0; y -= x)
    s += y;
  std::cout << "s=" << s << "\n";
  return 0;
}
```

Exercise 39 *Reformulate the code below equivalently in order to improve its readability. Describe the program's output as a function of its input* `n`. (G2)(G6)(G7)

```
unsigned int n;
std::cin >> n;
int x = 1;
if (n > 0) {
  int k = 0;
  bool e = true;
  do {
    if (++k == n) e = false;
    x *= 2;
  } while (e);
}
std::cout << x;
```

Exercise 40 *Reformulate the program below equivalently in order to improve its readability and efficiency. Describe the program's output as a function of its input* `x`. (G2)(G6)(G7)

```
#include <iostream>
int main()
{
  int x;
  std::cin >> x;
  int s = 0;
  int i = -10;
  do
    for (int j = 1;;)
      if (j++ < i) s += j - 1; else break;
  while (++i <= x);
  std::cout << s << "\n";
  return 0;
}
```

Exercise 41 *Write a program* `fak-1.C` *to compute the factorial* $n!$ *of a given input number* n. (G8)

Exercise 42 *Write a program* `dec2bin.C` *that inputs a natural number* n *and outputs the binary digits of* n *in reverse order. For example, for* $n==2$ *the output is* `01` *and for* $n==11$ *the output is* `1101` *(see also Exercise 45).* (G8)

Exercise 43 *Write a program* `cross_sum.C` *that inputs a natural number* n *and outputs the sum of the (decimal) digits of* n. *For example, for* $n==10$ *the output is* `1` *and for* $n==112$ *the output is* `4`. (G8)

Exercise 44 *Write a program* `perfect.C` *to test whether a given natural number* n *is perfect. A number* $n \in \mathbb{N}$ *is called* perfect *if and only if it is equal to the sum of its proper divisors, that is,* $n = \sum_{k\in\mathbb{N},\,\text{s.t.}\,k<n \wedge k|n} k$. *For example,* $28 = 1+2+4+7+14$ *is perfect, while* $12 < 1+2+3+4+6$ *is not.*

Extend the program to find all perfect numbers between 1 *and* n. *How many perfect numbers exist in the range* $[1, 50000]$*?* (G8)

Exercise 45 *Write a program* `dec2bin2.C` *that inputs a natural number* n *and outputs the binary digits of* n *in the correct order. For example, for* $n==2$ *the output is* `10` *and for* $n==11$ *the output is* `1011` *(see also Exercise 42).* (G8)

Exercise 46 *Pete and Colin play a dice game against each other. Pete has three four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4. Colin has two six-sided (cubical) dice, each with faces numbered 1, 2, 3, 4, 5, 6. Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal.*

What is the probability that Pyramidal Pete beats Cubic Colin? What is the probability that Cubic Colin beats Pyramidal Pete? And what is the probability of a draw? As a consequence, is it a fair game, and if not, who would you rather be?

Write a program `dice.C` *that outputs the aforementioned probabilities as rational numbers of the form* p/q. *(This is a simplified version of Problem 205 from the* Project Euler, *see* `http://projecteuler.net/`*.)* (G8)

Exercise 47 *We know from Section 1.1 that it took Frank Nelson Cole around three years to find the factorization*

$$761838257287 \cdot 193707721$$

of the Mersenne number $2^{67} - 1$ *by hand calculations. Write a program* `mersenne.C` *that performs the same task (hopefully in less than three years).* (G8)

Hint: *You will need the type* `ifm::integer`, *see Section 2.1.15.*

2.4.12 Challenges

Exercise 48 *The* n-queens problem *is to place* n *queens on an* $n \times n$ *chessboard such that no two queens threaten each other. Formally, this means that there is no horizontal, vertical, or diagonal with more than one queen in it. Write a program that outputs the number of different solutions to the* n*-queens problem for a given input* n. *Assuming a 32 bit system, the program should work up to* $n = 9$ *at least. Check through a web search whether the numbers that your program computes are correct.*

Exercise 49 *The largest Mersenne prime known as of September 2008 is*

$$2^{43,112,609} - 1.$$

In Exercise 17, we have asked you to find the number of decimal digits that this number has. In this challenge, we originally wanted to ask you to list all these digits, but in the interest of the TA that has to mark your solutions, we decided to switch to the following variant: Write a program `famous_last_digits.C` *that outputs the last 10 decimal digits of the above Mersenne prime!*

2.5 Floating point numbers

Furthermore, it has revealed the ratio of the chord and arc of ninety degrees, which is as seven to eight, and also the ratio of the diagonal and one side of a square which is as ten to seven, disclosing the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four.

Indiana House Bill No. 246, defining $\frac{2\sqrt{2}}{\pi} = \frac{7}{8}$, $\sqrt{2} = \frac{10}{7}$, *and* $\frac{1}{\pi} = 5/16$ *(1897)*

This section discusses the floating point number types `float` *and* `double` *for approximating real numbers. You will learn about floating point number systems in general, and about the IEEE standard 754 that describes two specific floating point number systems. We will point out the strengths and weaknesses of floating point numbers and give you three guidelines to avoid common pitfalls in computing with floating point numbers.*

When converting degrees Celsius into Fahrenheit with the program `fahrenheit.C` in Section 2.2, we make mistakes. For example, 28 degrees Celsius are 82.4 degrees Fahrenheit, but not 82 as output by `fahrenheit.C`. The reason for this mistake is that the integer division employed in the program simply “cuts off” the fractional part. What we need is a type that allows us to represent and compute with fractional numbers like 82.4.

For this, C++ provides two *floating point number* types `float` and `double`. Indeed, if we simply replace the declaration `int celsius` in `fahrenheit.C` by `float celsius`, the resulting program outputs `82.4` for an input value of 28. Floating point numbers also solve another problem that we had with the types `int` and `unsigned int`: `float` and `double` have a much larger value range and are therefore suitable for “serious” computations. In fact, computations with floating point numbers are very fast on modern platform, due to specialized processors.

Fixed versus floating point. If you think about how to represent decimal numbers like 82.4 using a fixed number of decimal digits (10 digits, say), a natural solution is this: you partition the 10 available digits into 7 digits before the decimal point, say, and 3 digits after the decimal point. Then you can represent all decimal numbers of the form

$$\sum_{i=-3}^{6} \beta_i 10^i,$$

with $\beta_i \in \{0, \ldots, 9\}$ for all i. This is called a *fixed point representation*.

There are, however, two obvious disadvantages of a fixed point representation. On the one hand, the value range is very limited. We have already seen in Section 2.2.5 that the largest `int` value is so small that it hardly allows any interesting computations (as an example, try out Program 1 on some larger input). A fixed point representation is even worse in this respect, since it reserves some of our precious digits for the fractional part after the decimal point, even if these digits are not—or not fully—needed (as in 82.4).

The second disadvantage is closely related: even though the two numbers 82.4 and 0.0824 have the same number of significant digits (namely 3), the latter number is not representable with only 3 digits after the decimal point. Here, we are wasting the 7 digits before the decimal point, but we are lacking digits after the decimal point.

A *floating point representation* resolves both issues by representing a number simply as its sequence of decimal digits (an integer called the *significand*), *plus* the information "where the decimal point is". Technically, one possibility to realize this is to store an *exponent* such that the represented number is of the form

$$\textit{significand} \cdot 10^{\textit{exponent}}.$$

For example,

$$\begin{aligned} 82.4 &= 824 \cdot 10^{-1}, \\ 0.0824 &= 824 \cdot 10^{-4}. \end{aligned}$$

2.5.1 The types float and double

The types `float` and `double` are fundamental types provided by C++, and they store numbers in floating point representation.

While the fundamental types `int` and `unsigned int` are meant to approximate the "mathematical types" $\mathbb{Z}$ and $\mathbb{N}$, respectively, the goal of both `float` and `double` is to approximate the set $\mathbb{R}$ of real numbers. Since there are much more real numbers than integers, this goal seems even more ambitious (and less realistic) than trying to approximate $\mathbb{Z}$, say, with a finite value range. Nevertheless, the two types `float` and `double` are very useful in practical applications. The floating point representation allows values that are much larger than any value of type `int` and `unsigned int`. In fact, the value ranges of the floating point number types `float` and `double` are sufficient in most applications.

Values of these two types are referred to as *floating point numbers*, where `double` usually allows higher (namely, *double*) precision in approximating real numbers.

On the types `float` and `double` we have the same arithmetic, relational, and assignment operators as on integral types, with the same associativities and precedences. The only exception is that the modulus operators `%` and `%=` are available for integral types only. This makes sense, since division over `float` and `double` is meant to model the true division over $\mathbb{R}$ which has no remainder.

Like integral types, the floating point number types are *arithmetic types*, and this completes the list of fundamental arithmetic types in C++.

Literals of type float and double. Literals of types `float` and `double` are more complicated than literals of type `int` or `unsigned int`. For example, `1.23e-7` is a valid `double` literal, representing the value $1.23 \cdot 10^{-7}$. Literals of type `float` look the same as literals of type `double`, followed by the letter `f` or `F`.

In its most general form, a `double` literal consists of an *integer part*, followed by a *fractional part* (starting with the *decimal point* `.`), and an *exponential part* (starting with the letter `e` or `E`). The literal `1.23e-7` has all of these parts.

Both the integer part as well as the fractional part (after the decimal point) are sequences of digits from `0` to `9`, where *one* of them may be empty, like in `.1` (meaning 0.1) and in `1.` (meaning 1.0). The exponential part (after the letter `e` or `E`) is also a sequence of digits, preceded by an optional `+` or `-`. *Either* the fractional part *or* the exponential part may be omitted. Thus, `123e-9` and `1.23` are valid `double` literals, but `123` is not, in order to avoid confusion with `int` literals.

The value of the literal is obtained by scaling the fractional decimal value defined by the integer part and the fractional part by 10^e, where e is the (signed) decimal integer in the exponential part (defined as 0, if the exponential part is missing).

To show floating point numbers in action, let us write a program that "computes" a fully-fledged real number, namely the Euler constant

$$\sum_{i=0}^{\infty} \frac{1}{i!} = 2.71828\ldots$$

You may recall that this sum converges quickly, so we should already get a good approximation for the Euler constant when we sum up the first 10 terms, say. Program 11 does exactly this.

```
// Program: euler.C
// Approximate Euler's constant e.

#include <iostream>

int main ()
{
  // values for term i, initialized for i = 0
  float t = 1.0f;    // 1/i!
  float e = 1.0f;    // i-th approximation of e

  std::cout << "Approximating the Euler constant...\n";
  // steps 1,...,n
  for (unsigned int i = 1; i < 10; ++i) {
```

```
15     e += t /= i;     // compact form of t = t / i; e = e + t
16     std::cout << "Value after term " << i << ": " << e << "\n";
17   }
18
19   return 0;
20 }
```

Program 11: *progs/euler.C*

When you run the program, its output may look like this.

```
Approximating the Euler constant...
Value after term 1: 2
Value after term 2: 2.5
Value after term 3: 2.66667
Value after term 4: 2.70833
Value after term 5: 2.71667
Value after term 6: 2.71806
Value after term 7: 2.71825
Value after term 8: 2.71828
Value after term 9: 2.71828
```

It seems that we do get a good approximation of the Euler constant in this way. What remains to be explained is how the mixed expression `e += t /= i` in line 15 is dealt with that contains operands of types `unsigned int` and `float`. Note that since the arithmetic assignment operators are right-associative (Table 1 on Page 48), this expression is implicitly parenthesized as `e += (t /= i)`. When evaluated in iteration i, it therefore first divides `t` by `i` (corresponding to the step from $1/(i-1)!$ to $1/i!$), and then it adds the resulting value $1/i!$ to the approximation `e`.

2.5.2 Mixed expressions, conversions, and promotions

The floating point number types are defined to be more general than any integral type. Thus, in mixed composite expressions, integral operands get converted to the respective floating point number type (see also Section 2.2.7 where we first saw this mechanism for mixed expressions over the types `int` and `unsigned int`). The resulting value is the representable value *nearest* to the original value. In particular, if the original integer value is in the value range of the relevant floating point number type, the value remains unchanged. If there are two nearest values, it is implementation-defined which one is chosen.

This in particular explains why the change of `int celsius` to `float celsius` in the program `fahrenheit.C` leads to the behavior we want: during evaluation of the expression `9 * celsius / 5 + 32`, all integral operands are eventually converted to `float`, so that the computation takes place exclusively over the type `float`.

In the program `euler.C`, we have the same kind of conversion: in the mixed expression `t /= i`, the `unsigned int` operand `i` gets converted to the type `float` of the other operand `t`.

The type `double` is defined to be more general than the type `float`. Thus, a composite expression involving operands of types `float` and `double` is of type `double`. When such an expression gets evaluated, any operand of type `float` is *promoted* to `double`. Recall from Section 2.3.2 that promotion is a term used to denote certain privileged conversions in which no information gets lost. In particular, the value range of `double` must contain the value range of `float`.

In summary, the hierarchy of arithmetic types from the least general to the most general type is

$$\texttt{bool} \prec \texttt{int} \prec \texttt{unsigned int} \prec \texttt{float} \prec \texttt{double}.$$

We already know that a conversion may also go from the more general to the less general type, see Section 2.2.7. This happens for example in the declaration statement

```
int i = -1.6f;
```

When a floating point number is converted to an integer, the fractional part is discarded. If the resulting value is in the value range of the target type, we get this value, otherwise the conversion is undefined. In the previous example, this rule initializes `i` with -1 (and *not* with the nearest representable value -2).

When `double` values are converted to `float`, we again get the nearest representable value (with ties broken in an implementation-dependent way), *unless* the original value is larger or smaller than any `float` value. In this latter case, the conversion is undefined.

2.5.3 Explicit conversions

Conversions between integral and floating point number types are common in practice. For example, the conversion of a nonnegative `float` value `x` to the type `unsigned int` corresponds to the well-known *floor function* $\lfloor x \rfloor$ that rounds down to the next integer. Conversely, it can make sense to perform an integral computation over a floating point number type, if this latter type has a larger value range.

Explicit conversion allows to convert a value of any arithmetic type directly into any other arithmetic type, without the detour of defining an extra variable like in `int i = -1.6f;` To obtain the `int` value resulting from the `float` value -1.6, we can simply write the expression `int(-1.6f)`.

The general syntax of an explicit conversion, also called a *cast expression*, is

T (*expr*)

where T is a type, and *expr* is an expression. The cast expression is valid if and only if the corresponding conversion of *expr* to the type T (as in `T x =` *expr*) is defined.

For certain "complicated" type names T, it is necessary to parenthesize T, like in the cast expression `(unsigned int)(1.6f)`.

2.5.4 Value range

For integral types, the arithmetic operations may fail to compute correct results only due to over- or underflow. This is because the value range of each integral type is a *contiguous* subset of $\mathbb{Z}$, with no "holes" in between.

For floating point number types, this is not true: with finite (and even with countable) value range, it is impossible to represent a subset of $\mathbb{R}$ with more than one element but no holes. In contrast, over- or underflows are less of an issue: the representable values usually span a huge interval, much larger than for integral types. If you print the largest `double` value on your platform via the expression

```
std::numeric_limits<double>::max()
```

you might for example get the output `1.79769e+308`. Recall that this means $1.79769 \cdot 10^{308}$, a pretty large number.

Let us approach the issue of holes with a very simple program that asks the user to input two floating point numbers *and* their difference. The program then checks whether this is indeed the correct difference. Program 12 performs this task.

```
// Program: diff.C
// Check subtraction of two floating point numbers

#include <iostream>

int main()
{
  // Input
  float n1;
  std::cout << "First number     =? ";
  std::cin >> n1;

  float n2;
  std::cout << "Second number    =? ";
  std::cin >> n2;

  float d;
  std::cout << "Their difference =? ";
  std::cin >> d;

  // Computation and output
  std::cout << "Computed difference - input difference = "
            << n1 - n2 - d << ".\n";
  return 0;
}
```

Program 12: *progs/diff.C*

Here is an example run showing that the authors are able to correctly subtract 1 from 1.5.

```
First number     =? 1.5
Second number    =? 1.0
Their difference =? 0.5
Computed difference - input difference = 0.
```

But the authors can apparently *not* correctly subtract 1 from 1.1:

```
First number     =? 1.1
Second number    =? 1.0
Their difference =? 0.1
Computed difference - input difference = 2.23517e-08.
```

What is going on here? After double checking our mental arithmetic, we must conclude that it's the *computer* and not us who cannot correctly subtract. To understand why, we have to take a somewhat closer look at floating point numbers in general.

2.5.5 Floating point number systems

A *finite floating point number system* is a finite subset of $\mathbb{R}$, defined by four numbers $2 \leq \beta \in \mathbb{N}$ (the *base*), $1 \leq p \in \mathbb{N}$ (the *precision*), $e_{\min} \in \mathbb{Z}$ (the *smallest exponent*) and $e_{\max} \in \mathbb{Z}$ (the *largest exponent*).

The set $\mathcal{F}(\beta, p, e_{\min}, e_{\max})$ of real numbers represented by this system consists of all floating point numbers of the form

$$s \cdot \sum_{i=0}^{p-1} d_i \beta^{-i} \cdot \beta^e,$$

where $s \in \{-1, 1\}$, $d_i \in \{0, \ldots, \beta - 1\}$ for all i, and $e \in \{e_{\min}, \ldots, e_{\max}\}$.

The number s is the *sign*, the sequence $d_0 d_1 \ldots d_{p-1}$ is called the *significand*[17], and the number e is the *exponent*.

We typically write a floating point number in the form

$$\pm d_0.d_1 \ldots d_{p-1} \cdot \beta^e.$$

For example, using base $\beta = 10$, the number 0.1 can be written as $1.0 \cdot 10^{-1}$, and as $0.1 \cdot 10^0$, $0.01 \cdot 10^1$ and in many other ways.

The representation of a number becomes unique when we restrict ourselves to the set $\mathcal{F}^*(\beta, p, e_{\min}, e_{\max})$ of *normalized* numbers, i.e. the ones with $d_0 \neq 0$. The downside of this is that we lose some numbers (in particular the number 0, but let's not worry about this now). More precisely, normalization loses exactly the numbers of absolute value smaller than $\beta^{e_{\min}}$ (see also Exercise 54).

[17] an older equivalent term is *mantissa*

For a fixed exponent e, the smallest positive normalized number is

$$1.0\ldots0 \cdot \beta^e = \beta^e,$$

while the largest one is[18]

$$(\beta-1).(\beta-1)\ldots(\beta-1)\cdot\beta^e = \sum_{i=0}^{p-1}(\beta-1)\beta^{-i}\cdot\beta^e = \left(1-\left(\frac{1}{\beta}\right)^p\right)\beta^{e+1} < \beta^{e+1}.$$

This means that the normalized numbers are "sorted by exponent".

Most floating point number systems used in practice are *binary*, meaning that they have base $\beta = 2$. In a binary system, the decimal numbers 1.1 and 0.1 are not representable, as we will see next; consequently, errors are made in converting them to floating point numbers, and this explains the strange behavior of Program 12.

Computing the floating point representation. In order to convert a given positive[19] decimal number x into a normalized binary floating point number system $\mathcal{F}^*(2, p, e_{\min}, e_{\max})$, we first compute its *binary expansion*

$$x = \sum_{i=-\infty}^{\infty} b_i 2^i, \quad b_i \in \{0,1\} \text{ for all } i.$$

This is similar to the binary expansion of a natural number as discussed in Section 2.2.8. The only difference is that we have to allow all negative powers of 2, since x can be arbitrarily close to 0. The binary expansion of 1.25 for example is

$$1.25 = 1 \cdot 2^{-2} + 1 \cdot 2^0.$$

We then determine the smallest and largest values of i, $\underline{i}$ and $\overline{i}$, for which b_i is nonzero (note that $\underline{i}$ may be $-\infty$, but $\overline{i}$ is finite since x is finite). The number $\overline{i}-\underline{i}+1 \in \mathbb{N}\cup\{\infty\}$ is the number of *significant digits* of x.

With $d_i := b_{\overline{i}-i}$, we get $d_0 \neq 0$ and

$$x = \sum_{i=\underline{i}}^{\overline{i}} b_i 2^i = \sum_{i=0}^{\overline{i}-\underline{i}} b_{\overline{i}-i} 2^{\overline{i}-i} = \sum_{i=0}^{\overline{i}-\underline{i}} d_i 2^{-i} \cdot 2^{\overline{i}}.$$

This implies that $x \in \mathcal{F}^*(2, p, e_{\min}, e_{\max})$ if and only if $\overline{i}-\underline{i} < p$ and $e_{\min} \leq \overline{i} \leq e_{\max}$. Equivalently, if the binary expansion of x has at most p significant digits, and the exponent of the normalized representation is within the allowable range.

In computing the binary expansion of $x > 0$, let us assume for simplicity that $x < 2$. This is sufficient to explain the issue with the decimal numbers 1.1 and 0.1, and all other

[18] by the formula $\sum_{i=0}^{n} x^i = (x^{n+1}-1)/(x-1)$ for $x \neq 1$

[19] Negative numbers are dealt with by the sign bit s.

cases can be reduced to this case by separately dealing with the largest even integer smaller or equal to x: writing $x = y + 2k$ with $k \in N$ and $y < 2$, we get the binary expansion of x by combining the expansions of y and $2k$.

For $x < 2$, we have

$$x = \sum_{i=-\infty}^{0} b_i 2^i = b_0 + \sum_{i=-\infty}^{-1} b_i 2^i = b_0 + \sum_{i=-\infty}^{0} b_{i-1} 2^{i-1} = b_0 + \frac{1}{2} \underbrace{\sum_{i=-\infty}^{0} b_{i-1} 2^i}_{=:x'}.$$

This identity provides a simple algorithm to compute the binary expansion of x. If $x \geq 1$, the most significant digit b_0 is 1, otherwise it is 0. The other digits b_i, $i \leq -1$, can subsequently be extracted by applying the same technique to $x' = 2(x - b_0)$.

Doing this for $x = 1.1$ yields the following sequence of digits.

$$\begin{array}{rclclcl}
 & & & & 1.1 & \rightarrow & b_0 = 1 \\
2(1.1-1) & = & 2 \cdot 0.1 & = & 0.2 & \rightarrow & b_{-1} = 0 \\
2(0.2-0) & = & 2 \cdot 0.2 & = & 0.4 & \rightarrow & b_{-2} = 0 \\
2(0.4-0) & = & 2 \cdot 0.4 & = & 0.8 & \rightarrow & b_{-3} = 0 \\
2(0.8-0) & = & 2 \cdot 0.8 & = & 1.6 & \rightarrow & b_{-4} = 1 \\
2(1.6-1) & = & 2 \cdot 0.6 & = & 1.2 & \rightarrow & b_{-5} = 1 \\
2(1.2-1) & = & 2 \cdot 0.2 & = & 0.4 & \rightarrow & b_{-6} = 0 \\
 & & & & & \vdots &
\end{array}$$

We now see that the binary expansion of the decimal number 1.1 is periodic: the corresponding binary number is $1.0\overline{0011}$, and it has infinitely many significant digits. Since all numbers in the floating point number systems $\mathcal{F}(2, p, e_{min}, e_{max})$ and $\mathcal{F}^*(2, p, e_{min}, e_{max})$ have at most p significant digits, it follows that $x = 1.1$ is not representable in a binary floating point number system, regardless of p, e_{min} and e_{max}. The same is true for $x = 0.1$.

The Excel 2007 bug. We have shown in the previous paragraph that it is impossible to convert some common decimal numbers (like 1.1 or 0.1) into binary floating-point numbers, without making small errors. This has the embarrassing consequence that the types `float` and `double` are unable to represent the values of some of their literals.

Despite this problem, a huge number of decimal-to-binary conversions take place on computers worldwide, the minute you read this. For example, whenever you enter a number into a spreadsheet, you do it in decimal format. But chances are high that internally, the number is converted to and represented in binary floating-point format. The small errors themselves are usually not the problem; but the resulting "weird" floating-point numbers extremely close to some "nice" decimal value may expose other problems in the program.

A recent such issue that has received a lot of attention is known as the the *Excel 2007 bug*. Users have reported that the multiplication of 77.1 with 850 in Microsoft Excel does not yield $65,535$ (the mathematically correct result) but $100,000$.

Microsoft reacted to this by admitting the bug, but at the same time pointing out that the *computed value* is correct, and that the error only happens when this value is *displayed* in the sheet. But how can it happen that the nice integer value 65,535 is incorrectly displayed? Well, it doesn't happen: when you multiply 65,535 with 1, for example, the result is correctly displayed as 65,535.

The point is that the computed value is *not* 65,535, but some other number extremely close to it. The reason is that a small but unavoidable error is made in converting the decimal value 77.1 into the floating-point number system internally used by Excel: like 1.1 and 0.1, the number 77.1 has no finite binary representation.

This error can of course not be "repaired" by the multiplication with 850, so Excel gets a value only very close to 65,535. This would be acceptable, but exactly for *this* value (and 11 others, according to Microsoft), the display functionality has a bug. Naturally, if only 12 "weird" numbers out of all floating-point numbers are affected by this bug, it is easy not to detect the bug during regular tests.

While Microsoft earned quite some ridicule for the Excel 2007 bug (for which it quickly offered a fix), it should in all fairness be admitted that such bugs could easily have occurred in software of other vendors as well.

Relative error. If we are not able to represent a real number x exactly as a binary floating point number in the system $\mathcal{F}^*(2, p, e_{\min}, e_{\max})$, it is natural to approximate it by the floating point number *nearest* to x. What is the error we make in this approximation?

Suppose that x is positive and has binary expansion

$$x = \sum_{i=-\infty}^{\bar{\imath}} b_i 2^i = b_{\bar{\imath}}.b_{\bar{\imath}-1} \ldots \cdot 2^{\bar{\imath}}, \quad \text{where } b_{\bar{\imath}} = 1.$$

There are two natural ways of approximating x with p or less significant digits. One way is to round down, resulting in the number

$$\underline{x} = b_{\bar{\imath}}.b_{\bar{\imath}-1} \ldots b_{\bar{\imath}-p+1} \cdot 2^{\bar{\imath}} = \sum_{i=\bar{\imath}-p+1}^{\bar{\imath}} b_i 2^i.$$

This truncates all the digits $b_i, i \leq \bar{\imath} - p$, and the error we make is

$$x - \underline{x} = \sum_{i=-\infty}^{\bar{\imath}-p} b_i 2^i \leq \sum_{i=-\infty}^{\bar{\imath}-p} 2^i = 2^{\bar{\imath}-p+1}.$$

Alternatively, we could round up to the number[20]

$$\overline{x} = \underline{x} + 2^{\bar{\imath}-p+1}$$

[20]We have to check that this number has at most p significant digits. This is true if $b_{\bar{\imath}-p+1} = 0$, since then the addition of $2^{\bar{\imath}-p+1}$ adds exactly one digit to the at most $p-1$ significant digits of $\underline{x}$. And if $b_{\bar{\imath}-p+1} = 1$, the addition of $2^{\bar{\imath}-p+1}$ removes the least significant coefficient of $2^{\bar{\imath}-p+1}$ and *may* create one extra carry digit at the other end.

where our previous error estimate shows that indeed, $x \leq \overline{x}$ holds.

This means that x is between two numbers that are $2^{\bar{i}-p+1}$ apart, so the nearer of the two numbers is at most $2^{\bar{i}-p}$ away from x. On the other hand, x has size *at least* $2^{\bar{i}}$, meaning that

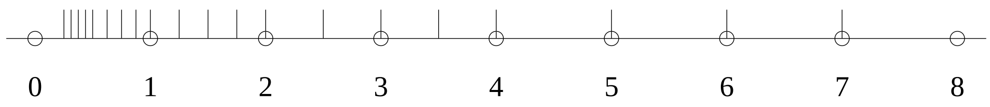
$$|x - \hat{x}|/x \leq 2^{-p},$$

where $\hat{x}$ is the floating point number nearest to x. The number 2^{-p}, referred to as the *machine epsilon*, is the *relative error* made in approximating x with its nearest floating point number $\hat{x}$.

The previous inequality also holds for negative x and their corresponding best approximation $\hat{x}$, so that we get the general *relative error* formula

$$\frac{|x - \hat{x}|}{|x|} \leq 2^{-p}, \quad x \neq 0.$$

This means that the distance of x to its nearest floating point number is in the worst case proportional to the size of x. This is because the floating point numbers are not equally spaced along the real line. Close to 0, their density is high, but the more we go away from 0, the sparser they become. As a simple example, consider the normalized floating point number system $F^*(2, 3, -2, 2)$. The smallest positive number is $1.00 \cdot 2^{-2} = 1/4$, and the largest one is $1.11 \cdot 2^2 = 7$ (recall that the digits are binary). The distribution of all positive numbers over the interval $[1/4, 7]$ is shown in the following picture.



From this picture, it is clear that the relative error formula cannot hold for very large x. But also if x is very close to zero, the relative error formula may fail. In fact, there is a substantial gap between 0 and the smallest positive normalized number. Numbers x in that gap are not necessarily approximable by normalized floating point numbers with relative error at most 2^{-p}.

Where is the mistake in our calculations, then? There is no mistake, but the calculations are only applicable if the floating point number $\hat{x}$ nearest to x *is* in fact a floating point number in the system we consider, i.e. if it has its exponent in the allowed range $\{e_{min}, \ldots, e_{max}\}$. This fails if $\hat{x}$ is too large or too small.

Arithmetic operations. Performing addition, subtraction, multiplication, and division with floating point numbers is easy in theory: as these are real numbers, we simply perform the arithmetic operations over the set $\mathbb{R}$ of real numbers; if the result is not representable in our floating point number system, we apply some rounding rule (such as choosing the nearest representable floating point number).

In practice, floating point number arithmetic is not more difficult than integer arithmetic. Let us illustrate this with an example. Suppose that $p = 4$, and that we have a binary system; we want to perform the addition

$$\begin{array}{rl} & 1.111 \cdot 2^{-2} \\ + & 1.011 \cdot 2^{-1} \end{array} .$$

The first step is to *align* the two numbers such that they have the same exponent. This means to "denormalize" one of the two numbers, e.g. the second one:

$$\begin{array}{rl} & 1.111 \cdot 2^{-2} \\ + & 10.110 \cdot 2^{-2} \end{array} .$$

Now we can simply add up the two significands, just like we add integers in binary representation. The result is

$$100.101 \cdot 2^{-2}.$$

Finally, we renormalize and obtain

$$1.00101 \cdot 2^{0}.$$

We now realize that this exact result is not representable with $p = 4$ significant digits, so we have to round. In this case, the nearest representable number is obtained by simply dropping the last two digits:

$$1.001 \cdot 2^{0}.$$

2.5.6 The IEEE standard 754

Value range. The C++ standard does not prescribe the value range of the types `float` and `double`. It only stipulates that the value range of `float` is contained in the value range of `double` such that a `float` value can be promoted to a `double` value.

In practice, most platforms support (variants of) the *IEEE standard 754* for representing and computing with floating point numbers. Under this standard, the value range of the type `float` is the set

$$F^*(2, 24, -126, 127)$$

of *single precision* normalized floating point numbers, plus some special numbers (conveniently, 0 is one of these special numbers). The value range of `double` is the set

$$F^*(2, 53, -1022, 1023)$$

of *double precision* normalized floating point numbers, again with some special numbers added, including 0.

These parameters may seem somewhat arbitrary at first, but they are motivated by a common memory layout in which 32 bits form a memory cell. Indeed, 32 bits of memory are used to represent a single precision number. The significand requires 23 bits; recall that in a normalized binary floating point number system, the first digit of the significand is always 1, hence it need not explicitly be stored. The exponent requires another 8 bits for representing its $254 = 2^8 - 2$ possible values, and another bit is needed for the sign.

For double precision numbers, the significand requires 52 bits, the exponent has 11 bits for its $2046 = 2^{11} - 2$ possible values, and one bit is needed for the sign. In total, this gives 64 bits.

Note that in both cases, two more exponent values could be accommodated without increasing the total number of bits. These extra values are in fact used for representing the special numbers mentioned above, including 0.

Requirements for the arithmetic operations. The C++ standard does not prescribe the accuracy of arithmetic operations over the types `float` and `double`, but the IEEE standard 754 does. The requirements are as strict as possible: the result of any addition, subtraction, multiplication, or division is the representable value *nearest* to the true value. If there are two nearest values (meaning that the true value is halfway in between them), the one that has least significant digit $d_{p-1} = 0$ is chosen.[21] The same rule applies to the conversion of decimal values like 1.1 to their binary floating point representation.

Moreover, comparisons of values have to be exact under all relational operators (Section 2.3.2).

2.5.7 Computing with floating point numbers

We have seen that for any floating point number system, there are numbers that it cannot represent, and these are not necessarily very exotic, as our example with the decimal number 1.1 shows. On the other hand, the IEEE standard 754 guarantees that we will get the nearest representable number, and the same holds for the result of any arithmetic operation, up to (rare) over- and underflows. Given this, one might be tempted to believe that the results of most computations involving floating point numbers are close to the mathematically correct results, with respect to relative error.

Indeed, this is true in many cases. For example, our initial program `euler.C` computes a pretty good approximation of the Euler constant. Nevertheless, some care has to be taken in general. The goal of this section is to point out common pitfalls, and to provide resulting guidelines for "safe" computations with floating point numbers.

We start with the first and most important guideline that may already be obvious to you at this point.

> **Floating Point Arithmetic Guideline 1:** Never compare two floating point numbers for equality, if at least one of them results from inexact floating point computations.

[21] this is called *round-to-even*; other rounding modes can be enabled if necessary.

Even very simple expressions involving floating point numbers may be mathematically equivalent, but still return different values, since intermediate results are rounded. Two such expressions are `x * x - y * y` and `(x + y) * (x - y)`. Therefore, testing the results of two floating point computations for equality using the relational operators `==` or `!=` makes little sense. Since equality is sensitive to the tiniest errors, we won't get equality in most cases, even if mathematically, we would.

Given the formulation of the above guideline, you may wonder how to tell whether a particular floating point computation is exact or not. Exactness usually depends on the representation and is, therefore, hard to claim in general. However, there are certain operations which are easily seen to be exact. For instance, multiplication and division by a power of the base (usually, 2) do not change the significand, but only the exponent. Thus, these operations are exact, unless they lead to an over- or underflow in the exponent.

Moreover, it is safe to assume that the largest exponent is (much) higher than the precision p, and in this case we can also exactly represent all integers of absolute value smaller than β^p. Consequently, integer additions, subtractions, and multiplications within this range are exact.

The next two guidelines are somewhat less obvious, and we motivate them by first showing the underlying problem. Throughout, we assume a binary floating point number system of precision p.

Adding numbers of different sizes. Suppose we want to add the two floating point numbers 2^p and 1. What will be the result? Mathematically, it is

$$2^p + 1 = \sum_{i=0}^{p} b_i 2^i,$$

with $(b_p, b_{p-1}, \ldots, b_0) = (1, 0, \ldots, 0, 1)$. Since this binary expansion has $p+1$ significant digits, $2^p + 1$ is not representable with precision p. Under the IEEE standard 754, the result of the addition is 2^p (chosen from the two nearest candidates 2^p and $2^p + 2$), so this addition has no effect.

The general phenomenon here is that adding floating point numbers of different sizes "kills" the less significant digits of the smaller number (in our example, *all* its digits). The larger the size difference, the more drastic is the effect.

To convince you that this is not an artificial phenomenon, let us consider the problem of computing Harmonic numbers (search the web for the *coupon collector's problem* to find an interesting occurrence of Harmonic numbers). For $n \in \mathbb{N}$, the n-th *Harmonic number* H_n is defined as the sum of the reciprocals of the first n natural numbers, that is,

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

It should now be an easy exercise for you to write a program that computes H_n for a given $n \in \mathbb{N}$. You only need a single loop running through the numbers 1 up to n, adding their reciprocals. Just as well you can make your loop run from n down to 1 and sum up the reciprocals. Why not, that should not make any difference, right? Let us try both variants and see what we get. The program `harmonic.C` shown below computes the two sums and outputs them.

```
1  // Program: harmonic.C
2  // Compute the n-th harmonic number in two ways.
3
4  #include <iostream>
5
6  int main()
7  {
8    // Input
9    std::cout << "Compute H_n for n =? ";
10   unsigned int n;
11   std::cin >> n;
12
13   // Forward sum
14   float fs = 0;
15   for (unsigned int i = 1; i <= n; ++i)
16     fs += 1.0f / i;
17
18   // Backward sum
19   float bs = 0;
20   for (unsigned int i = n; i >= 1; --i)
21     bs += 1.0f / i;
22
23   // Output
24   std::cout << "Forward sum  = " << fs << "\n"
25             << "Backward sum = " << bs << "\n";
26   return 0;
27 }
```

Program 13: *progs/harmonic.C*

Think for a second and recall why it is important to *not* write `1 / i` in line 16 and line 21. Now let us have a look at an execution of the program.

```
Compute H_n for n =? 10000000
Forward sum  = 15.4037
Backward sum = 16.686
```

The results differ significantly. The difference becomes even more apparent when we try larger inputs.

```
Compute H_n for n =? 100000000
```

```
Forward sum  = 15.4037
Backward sum = 18.8079
```

Notice that the forward sum did not change, which cannot be correct. Using the approximation

$$\frac{1}{2(n+1)} < H_n - \ln n - \gamma < \frac{1}{2n},$$

where $\gamma = 0.57721666\ldots$ is the Euler-Mascheroni constant, we get $H_n \approx 18.998$ for $n = 10^8$. That is, the backward sum provides a much better approximation of H_n.

Why does the forward sum behave so badly? The reason is simple: As the larger summands are added up first, the intermediate value of the sum to be computed grows (comparatively) fast. At some point, the size difference between the partial sum and the summand $\frac{1}{i}$ to be added is so large that the addition does not change the partial sum anymore, just like in $2^p + 1$ "$=$" 2^p. Thus, regardless of how many more summands are added to it, the sum stays the same.

In contrast, the backward sum starts to add up the small summands first. Therefore, the value of the partial sum grows (comparatively) slowly, allowing the small summands to contribute. The summands treated in the end of the summation have still a good chance to influence the significand of the partial sum, since they are (comparatively) large.

The phenomenon just observed leads us to our second guideline.

Floating Point Arithmetic Guideline 2: Avoid adding two numbers that considerably differ in size.

Cancellation. Consider the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

It is well known that its two roots are given by

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In a program that computes these roots, we might therefore want to compute the value $d = b^2 - 4ac$ of the *discriminant*. If b^2 and $4ac$ are representable as floating point numbers with precision p, our previous error estimates guarantee that the result $\hat{d}$ of the final subtraction has small relative error: $|d - \hat{d}| \leq 2^{-p}|d|$. This means, even if d is close to zero, $\hat{d}$ will be away from d by *much less* than the distance of d to zero.

The problem arises if the numbers b^2 and/or $4ac$ are not representable as floating point numbers, in which case errors are made in computing them. Assume $b = 2^p$, $a = 2^{p-1} - 1$, $c = 2^{p-1} + 1$ (all these numbers are exactly representable). Then the exact value of d is 4. The value $b^2 = 2^{2p}$ is a representable floating point number, but $4ac = 2^{2p} - 4$ is not, since this number has $2p-2$ significant digits (all of them equal to 1) in its binary expansion. The nearest floating point number is obtained by rounding up (adding 4), and after the (error-free) subtraction, we get $\hat{d} = 0$. The relative error of this computation is therefore 1 instead of 2^{-p}.

The reason is that in subtracting two numbers that are almost equal, the more significant digits cancel each other. If, on the other hand, the remaining less significant digits already carry some errors from previous computations, the subtraction hugely amplifies these errors: the cancellation promotes the previously less significant digits to much more significant digits of the result.

Again, the example we gave here is artificial, but be assured that cancellation happens in practice. Even in the quadratic equation example, it might be that the equations that come up in an application have the property that their discriminant $b^2 - 4ac$ is much smaller than a, b and c themselves. In this case, cancellation *will* happen.

The discussion can be summarized in form of a third guideline.

Floating Point Arithmetic Guideline 3: Avoid subtracting two numbers of almost equal size, if these numbers are results of other floating point computations.

2.5.8 Details

Other floating point number systems. The IEEE standard 754 defines two more floating point number systems, *single-extended precision* ($p = 32$), and *double-extended precision* ($p = 64$), and some platforms offer implementations of these types. There is also the IEEE standard 854 that allows base $\beta = 10$, for obvious reasons: the decimal format is the one in which we think about numbers, and in which we usually represent numbers. In particular, a base-10 system has no holes in the value range at decimal fractional numbers like 1.1 and 0.1.

IEEE compliance. While on most platforms, the types `float` and `double` correspond to the single and double precision floating point numbers of the IEEE standard 754, this correspondence is usually not one-to-one. For example, if you are trying to reproduce the cancellation example we gave, you might write

```
float b = 16777216.0f;  // 2^24
float a =  8388607.0f;  // 2^23 - 1
float c =  8388609.0f;  // 2^23 + 1

std::cout << b * b - 4.0f * a * c << "\n";
```

and expect to get the predicted wrong result 0. But it may easily happen that you get the correct result 4, even though your platform claims to follow the IEEE standard 754. The most likely reason is that the platform internally uses a register with more bits to perform the computation. While this seems like a good idea in general, it can be fatal for a program whose functionality critically relies on the IEEE standard 754.

You *will* most likely see the cancellation effect in the following seemingly equivalent variant of the above code.

```
float b = 16777216.0f;  // 2^24
float a =  8388607.0f;  // 2^23 - 1
float c =  8388609.0f;  // 2^23 + 1

float bb = b * b;
float ac4 = 4.0f * a * c;

std::cout << bb - ac4 << "\n";
```

Here, the results of the intermediate computations are written back to `float` variables, probably resulting in the expected rounding of $4ac$. Then the final subtraction reveals the cancellation effect. *Unless*, of course, the compiler decides to keep the variable `ac4` in a register with more precision. For this reason, you can typically provide a compiler option to make sure that floating point numbers are not kept in registers.

What is the morale of this? You usually cannot fully trust the *IEEE compliance* of a platform, and it is neither easy nor worthwhile to predict how floating point numbers exactly behave on a specific platform. It is more important for you to know and understand floating point number systems in general, along with their limitations. This knowledge will allow you to identify and work around problems that might come up on specific platforms.

The type long double. The C++ standard prescribes another fundamental floating point number type called `long double`. Its literals end with the letter `l` or `L`, and it is guaranteed that the value range of `double` is contained in the value range of `long double`. Despite this, the conversion from `double` to `long double` is not defined to be a promotion by the C++ standard.

While `float` and `double` usually correspond to single and double precision of the IEEE standard 754, there is no such default choice for `long double`. In practice, `long double` might simply be a synonym for `double`, but it might also be something else. On the platform used by the authors, for example, `long double` corresponds to the normalized floating point number system $F^*(2, 64, -16382, 16384)$— this is exactly the double-extended precision of the IEEE standard 754.

Numeric limits. If you want to know the parameters of the floating point number systems behind `float`, `double` and `long double` on your platform, you can employ the `numeric_limits` we have used before in the program `limits.C` in Section 2.2.5. Here are the relevant expressions together with their meanings, shown for the type `float`.

expression (of type int)	meaning
std::numeric_limits<float>::radix	β
std::numeric_limits<float>::digits	p
std::numeric_limits<float>::min_exponent	$e_{\min} + 1$
std::numeric_limits<float>::max_exponent	$e_{\max} + 1$

We remark that std::numeric_limits<float>::min() does *not* give the smallest float value (because of the sign bit, this smallest value is simply the negative of the largest value), but the smallest normalized *positive* value.

Special numbers. We have mentioned that the floating point systems prescribed by the IEEE standard 754 contain some special numbers; their encoding uses exponent values that do not occur in normalized numbers.

On the one hand, there are the *denormalized* numbers of the form

$$\pm d_0.d_1 \ldots d_{p-1} \cdot \beta^{e_{\min}},$$

with $d_0 = 0$. A denormalized number has smaller absolute value than any normalized number. In particular, 0 is a denormalized number.

The other special numbers cannot really be called numbers. There are values representing $+\infty$ and $-\infty$, and they are returned by overflowing operations. Then there are several values called NaNs (for "not a number") that are returned by operations with undefined result, like taking the square root of a negative number. The idea behind these values is to provide more flexibility in dealing with exceptional situations. Instead of simply aborting the program when some operation fails, it makes sense to return an exceptional value. The caller of the operation can then decide how to deal with the situation.

2.5.9 Goals

Dispositional. At this point, you should ...

1) know the floating point number types float and double, and that they are more general than the integral types;
2) understand the concept of a floating point number system, and in particular its advantages over a fixed point number system;
3) know that the IEEE standard 754 describes specific floating point number systems used as models for float and double on many platforms;
4) know the three Floating Point Arithmetic Guidelines;
5) be aware that computations involving the types float and double may deliver inexact results, mostly due to holes in the value range.

Operational. In particular, you should be able to ...

(G1) evaluate expressions involving the arithmetic types `int`, `unsigned int`, `float` and `double`;

(G2) compute the binary representation of a given real number;

(G3) compute the floating point number nearest to a given real number, with respect to a finite floating point number system;

(G4) work with a given floating point number system;

(G5) recognize usage of floating point numbers that violates any of the three Floating Point Arithmetic Guidelines;

(G6) write programs that perform computations with floating point numbers.

2.5.10 Exercises

Exercise 50 *Evaluate the following expressions step-by-step, according to the conversion rules of mixed expressions. We assume a floating point representation according to IEEE 754, that is,* `float` *corresponds to* $F^*(2, 24, -126, 127)$ *and double corresponds to* $F^*(2, 53, -1022, 1023)$*. We also assume that 32 bits are used to represent* `int` *values.* *(G1)*

a) `6 / 4 * 2.0f - 3`

b) `2 + 15.0e7f - 3 / 2.0 * 1.0e8`

c) `392593 * 2735.0f - 8192 * 131072 + 1.0`

d) `16 * (0.2f + 262144 - 262144.0)`

Exercise 51 *Compute the binary expansions of the following decimal numbers.*
a) 0.25 *b)* 1.52 *c)* 1.3 *d)* 11.1 *(G2)*

Exercise 52 *For the numbers in Exercise 51, compute nearest floating point numbers in the systems* $\mathcal{F}^*(2, 5, -1, 2)$ *and* $\mathcal{F}(2, 5, -1, 2)$*.* *(G3)*

Exercise 53 *What are the largest and smallest* positive *normalized single and double precision floating point numbers, according to the IEEE standard 754?* *(G4)*

Exercise 54 *How many floating point numbers do the systems* $\mathcal{F}^*(\beta, p, e_{min}, e_{max})$ *and* $\mathcal{F}(\beta, p, e_{min}, e_{max})$ *contain?* *(G4)*

Exercise 55 *Compute the value of the variable* `d` *after the declaration statement*

```
float d = 0.1;
```

Assume the IEEE standard 754. *(G3)*

Exercise 56 *What is the (potential) problem with the following loop?* (G5)

```
for (float i = 0.1; i != 1.0; i += 0.1)
  std::cout << i << "\n";
```

Exercise 57 *What is the (potential) problem with the following loop?* (G5)

```
for (float i = 0.0f; i < 100000000.0f; ++i)
    std::cout << i << "\n";
```

Exercise 58 *Write a program that outputs for a given decimal input number* x, $0 < x < 2$, *its normalized* `float` *value on your platform. The output should contain the (binary) digits of the significand, starting with* 1, *and the (decimal) exponent. You may assume that the floating point number system underlying the type* `float` *has base* $\beta = 2$. *(G3)(G6)*

Exercise 59 *Write a program that tests whether a given value of type* `double` *is actually an integer, and test the program with various inputs like* `0.5`, `1`, `1234567890`, `1234567890.2`. *Simply converting to a value of type* `int` *and checking whether this changes the value does not work in general, since the given value might be an integer outside the value range of* `int`. *You may assume that the floating point number system underlying the type* `double` *has base* $\beta = 2$. *(G3)(G6)*

Exercise 60 *The number* π *can be defined through various infinite sums. Here are two of them.*

$$\begin{aligned}\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ \frac{\pi}{2} &= 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots\end{aligned}$$

Write a program for computing an approximation of π, *based on these formulas. Which formula is better for that purpose?* *(G6)*

Exercise 61 *There is a well-known iterative procedure (the* Babylonian method*) for computing the square root of a positive real number* s. *Starting from any value* $x_0 > 0$, *we compute a sequence* $x_0, x_1`, x_2, \ldots$ *of values according to the formula*

$$x_n = \frac{1}{2}(x_{n-1} + \frac{s}{x_{n-1}}).$$

It can be shown that

$$\lim_{n\to\infty} x_n = \sqrt{s}.$$

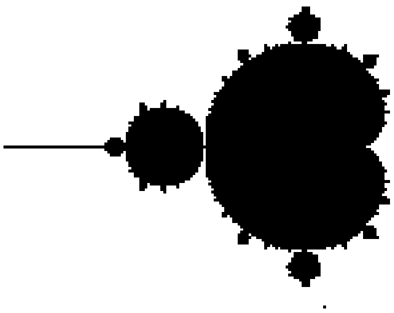
Write a program `babylonian.C` *that reads in the number* s *and computes an approximation of* $\sqrt{s}$ *using the Babylonian method. To be concrete, the program should output the first number* x_i *such that* *(G6)*

$$|x_i^2 - s| < 0.001.$$

Exercise 62 *Write a program* `fpsys.C` *to visualize a normalized floating point number system* $\mathcal{F}^*(2, p, e_{\min}, e_{\max})$*. The program should read the parameters* p*,* $e_{\min}$*, and* $e_{\max}$ *as inputs and for each positive number* x *from* $\mathcal{F}^*(2, p, e_{\min}, e_{\max})$ *draw a circle of radius* x *around the origin. Use the library* `libwindow` *that is available at the course homepage to create graphical output. Use the program to verify the numbers you computed in Exercise 54.* *(G4)(G6)*

2.5.11 Challenges

Exercise 63 *The* Mandelbrot set *is a subset of the complex plane that became popular through its* fractal *shape and the beautiful drawings of it. Below you see the set's* main cardioid *and a detail of it at much higher zoom scale.*



The Mandelbrot set is defined as follows. For $c \in \mathbb{C}$*, we consider the sequence* $z_0(c), z_1(c), \ldots$ *of complex numbers given by* $z_0(c) = 0$ *and*

$$z_n(c) = z_{n-1}(c)^2 + c, \quad n > 0.$$

There are two cases: either $|z_n(c)| \leq 2$ *for all* n *(this obviously happens for example if* $c = 0$*), or* $|z_n(c)| > 2$ *for some* n *(this obviously happens for example if* $|c| > 2$*). The Mandelbrot set consists of all* c *for which we are in the first case. It follows that the Mandelbrot set contains* 0 *and is contained in a disk of radius* 2 *around* 0 *in the complex plane.*

Write a program that draws (an approximation of) the Mandelbrot set, restricted to a rectangular subset of the complex plane. It should be possible to zoom in, meaning that the rectangular subset becomes smaller, and more details become visible in the drawing window. Obviously, you can't process all infinitely many complex numbers c *in the rectangle, and for given* c*, you cannot really check whether* $|z_n(c)| \leq 2$ *for* all n*, so it is necessary to discretize the rectangle into pixels, and to establish some upper bound* N *on the number of iterations. If* $|z_n(c)| \leq 2$ *for all* $n \leq N$*, you may simply assume that* c *is in the Mandelbrot set. Per se, there is no guarantee that the resulting drawing is even close to the Mandelbrot set (especially at finer level of detail), but for the sake of obtaining nice pictures, we can generously gloss over this issue.*

Hint: *You may use the* `libwindow` *library to produce the drawing. The example program in its documentation should give you an idea how this can be done.*

Exercise 64 *The following email was sent to a mailing list for users of the software library CGAL.*

```
Hi all,

This should be a very easy question.

When I check if the points (0.14, 0.22), (0.15, 0.21) and (0.19,0.17) are
collinear, using CGAL::orientation, it returns CGAL::LEFT_TURN, which is
false, because those points are in fact collinear.

However, if I do the same with the points (14, 22), (15, 21) and (19, 17) I
get the correct answer: CGAL::COLLINEAR.
```

a) *Find out what this email is about; in particular, what is CGAL, what is the orientation of a point triple, what is* `CGAL::orientation`*, what does "collinear" mean, and why is the writer of the email surprised about the observed behavior?*

b) *Draft an answer to this email that explains the observations of the CGAL user that wrote it.*

2.6 Arrays and pointers

Reading into an array without making a "silly error" is beyond the ability of complete novices - by the time you get that right, you are no longer a complete novice.

Bjarne Stroustrup, C++ Style and Technique FAQ

This section introduces arrays as containers for sequences of objects of the same type, with random access to individual members of the sequence. An array is the most primitive but at the same time a very efficient container for storing, processing, and iterating over large amounts of data. You will also learn about pointers as explicit object addresses and about their close relationship with arrays. While the C++ standard library contains less primitive and generally better alternatives, the concepts *behind arrays and pointers are of fundamental importance.*

In Section 2.4 on control statements, we have learned about the concept of iteration. For example, we can now iterate over the sequence of numbers $1, 2, \ldots, n$ and perform some operations like adding up all the numbers, or identifying the prime numbers among them. Similarly, we can iterate over the odd numbers, the powers of two, etc.

In real applications, however, we often have to process (and in particular iterate over) sequences of *data*. For example, if you want to identify the movie theaters in town that show your desired movie tonight, you have to iterate over the sequence of movie theater repertoires. These repertoires must be stored somewhere, and there must be a way to inspect them in turn. In C++, we can deal with such tasks by using *arrays*.

2.6.1 Array types

An array of length n aggregates n objects of the *same* type T into a sequence. To access one of the aggregated objects (the *elements*), we use its *index* or *subscript* (position) in the sequence. All these length-n sequences form an array type whose value range corresponds to the mathematical type T^n. In the computer's main memory, an array occupies a contiguous part, with the elements stored side-by-side (see Figure 6).

Let us start by showing an array in action: *Eratosthenes' Sieve* is a fast method for computing all prime numbers smaller than a given number n, based on crossing out the numbers that are not prime. It works like this: you write down the sequence of numbers between 2 and $n-1$. Starting from 2, you always go to the next number not crossed out yet, report it as prime, and then cross out all its proper multiples.

Let's not dwell on the correctness of this method but go right to the implementation. If you think about it for a minute, the major question is this: how do we cross out numbers?

The following program uses an array type variable `crossed_out` for the list, where any value `crossed_out[i]` is of type `bool` and represents the (changing) information whether the number i has already been crossed out or not. Array indices always start from 0, so in order to get to index $n-1$, we need an array of length n. The program runs Eratosthenes' Sieve for $n = 1,000$.

```
// Program: eratosthenes.C
// Calculate prime numbers in {2,...,999} using
// Eratosthenes' sieve.

#include <iostream>

int main()
{
  // definition and initialization: provides us with
  // Booleans crossed_out[0],..., crossed_out[999]
  bool crossed_out[1000];
  for (unsigned int i = 0; i < 1000; ++i)
    crossed_out[i] = false;

  // computation and output
  std::cout << "Prime numbers in {2,...,999}:\n";
  for (unsigned int i = 2; i < 1000; ++i)
    if (!crossed_out[i]) {
      // i is prime
      std::cout << i << " ";
      // cross out all proper multiples of i
      for (unsigned int m = 2*i; m < 1000; m += i)
        crossed_out[m] = true;
    }
  std::cout << "\n";

  return 0;
}
```

Program 14: *progs/eratosthenes.C*

Definition. An array variable (or simply array) a with $n > 0$ elements of *underlying type* T is defined through the following declaration.

T a[*expr*]

Here, *expr* must be a *constant expression* of integral type whose value is n. For example, literals like `1000`, or arithmetic expressions over literals (like `1+1`) are constant expressions; there are other constant expressions, but all of them have the property that their value is known at compile time. This allows the compiler to figure out how much memory the array variable needs.

The type of *a* is "$T[n]$", but we put this in double quotes here (only to omit them later). The reason is that $T[n]$ is not the official name: we can't write `int[5] a`, for example, to declare an array `a` of type `int[5]`.

The value range of $T[n]$ is T^n, the set of all sequences $(t_1, t_2, \ldots, t_n)$ with all t_i being of type T. The underlying type T might for example be any fundamental type (like `int`, `bool`, or `double`), and in this case, the values of the n array elements remain uninitialized by the definition.

The fact that the array length must be known at compile time clearly limits the usefulness of array variables. For example, this limitation does not allow us to write a version of Eratosthenes' sieve in which the number n is read from the input. But we will shortly see how this restriction can be overcome—for the time being, let's simply live with it.

2.6.2 Initializing arrays

The definition of an array with underlying fundamental type does not initialize the values of the array elements. We can assign values to the elements afterwards (like we do it in Program 14), but we can also provide the values directly, as in the following declaration statement.

```
int a[5] = {4,3,5,2,1};
```

Since the number of array elements can be deduced from the length of the *initializer list*, we can also write

```
int a[] = {4,3,5,2,1};
```

The declaration `int a[]` without any initialization is invalid, though, since it does not fully determine the type of `a`. We say that `a` has *incomplete type* in this case.

2.6.3 Random access to elements

The most common and useful way of accessing and modifying the elements of an array is by *random access*. If *expr* is of integral type and has value i, the lvalue

a[*expr*]

is of the type underlying the array `a` and refers to the i-th element (counting from 0) of `a`. The number i is called the *index* or *subscript* of the element. If n is the length of `a`, the index i must satisfy $0 \leq i < n$. The operator `[]` is called the *subscript operator*.

The somewhat strange declaration format of an array, with no explicit type name appearing, is motivated by the subscript operator. Indeed, the declaration

T a[*expr*]

can be read as "*a*[*expr*] is of type *T*". In this sense, it is an implicit definition of a's type.

> **Watch out!** The C++ language offers no functionality for accessing the length of an array (see Section 2.6.4 below for more on this). As the programmer, *you* must remember the length yourself, and *you* are responsible for making sure that a given array index i indeed satisfies $0 \leq i < n$, where n is the length of the array. Indices that are not in this range are called *out of bound*. Unless your compiler offers specific debugging facilities, the usage of out-of-bound indices in the subscript operator is *not* detected at runtime and leads to undefined behavior of the program.

We have already discussed the term random access in connection with the computer's main memory (Section 1.2.3); random access means that *every* array element can be accessed in the same uniform way, and with (almost) the same access time, no matter what its index is. Evaluating the expression `a[0]` is as fast as evaluating `a[10000]`. In contrast, the thick pile of pending invoices, bank transfers and various other papers on your desk does not support random access: the time to find an item is roughly proportional to its depth within the pile.

In fact, random access in an array directly reduces to random access in the computer's main memory, since an array always occupies a contiguous set of memory cells, see Figure 6.

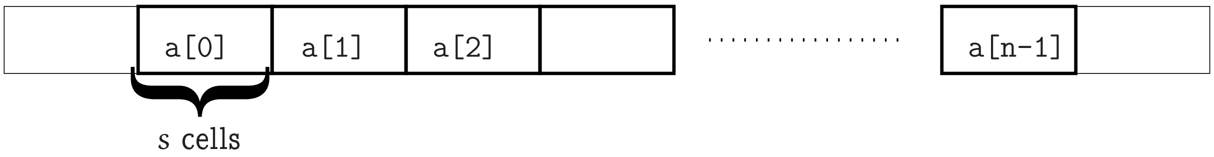


Figure 6: *An array occupies a contiguous part of the main memory. Every element in turn occupies s memory cells, where s is the memory required to store a single value of the underlying type T.*

To access the element of index i in the array a, a simple computation with addresses therefore suffices. If p is the address (position) where the first element of a "starts", and s is the number of memory cells that a single value of the underlying type T occupies, then the element of index i starts at the memory cell whose address is $p+si$, see Figure 7.

2.6.4 Arrays are not self-describing

Array types are exceptional in C++. The following code fragment illustrates this:

```
int a[5] = {4,3,5,2,1}; // array of type int[5]
int b[5];
b = a;                  // error: we cannot assign to an array
```

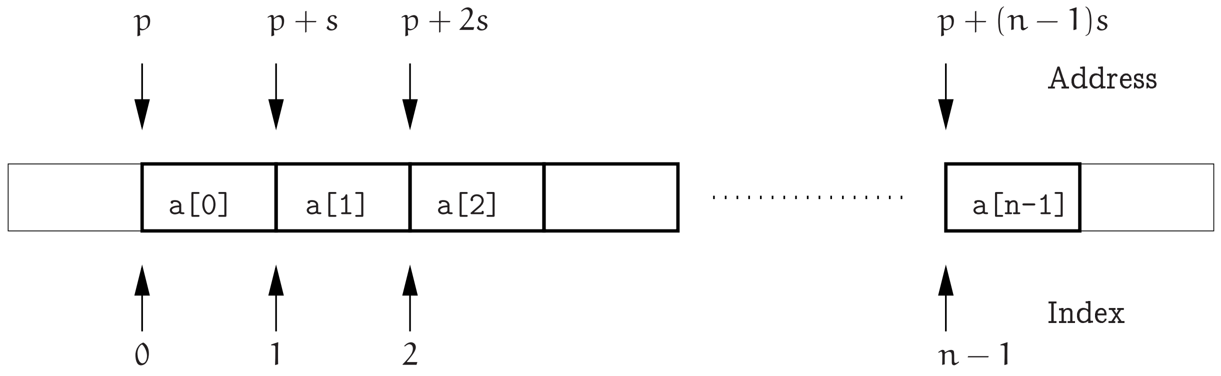


Figure 7: *The array element of index* i *starts at the address* $p + si$.

We also cannot initialize an array from another array. Why is this? Arrays are a dead hand from the programming language C, and the design of arrays in C is (from today's point of view) quite primitive. The main weakness is that *the number of elements of an array is not represented in the array's value*. C was designed to compete with machine language in efficiency, and this didn't leave room for luxury. An array variable is merely represented by its address, defined as the address of the memory cell where the first element (the one of index 0) starts. We also say that an array is not *self-describing*: it does not "know" its own length.

Now we get to the assignment issue: there is no way of automatically copying all elements of the array a into b, without knowing how many elements these are. But we *can* of course do this manually via a simple loop, since we know the array lengths from the declarations; early C programmers were not yet spoiled enough to complain about such minor inconveniences. (On the other hand, this leaves ample room for bugs.)

When C++ was developed much later, one design goal was to have C as a subset. As a consequence, arrays are still around in C++.

2.6.5 Iteration over a container

Let's take a step back, forget about the technicalities of arrays for a moment, and go for a bigger picture.

We have already indicated in the introduction to this section that the process of iterating over a sequence of data is ubiquitous. Typically, the data are stored in some *container*, and we need to perform a certain operation for all elements in the container. In general, a container is an object that can store other objects (its elements), and that offers some ways of accessing these elements. The only "hard" requirement here is that a container must offer the possibility of iterating over all its elements. In this informal sense, an array is indeed a container, since the random access functionality can be used to iterate over the elements.

Iteration by random access. Let's get back to arrays. Iterating over an array of length n can be done by random access like in lines 12–13 of Program 14. We have seen that the random access functionality of arrays is internally based on address arithmetic. During the iteration, the following sequence of addresses is computed: $p, p+s, p+2s, \ldots, p+(n-1)s$, where p and s have the usual meanings.

This requires one multiplication and one addition for any address except the first. But if you think about it, the multiplication only comes in because we compute each address from scratch, independently from the previous ones. In fact, the same set of addresses could more efficiently and more naturally be computed by starting with p and repeatedly adding s ("going to the next element").

Using random access, we can *simulate* array iteration, but we are missing the operation of "going to the next element"; only this operation makes iteration over a container natural and efficient. The following analogy illustrates the point: you *can* of course read a book by starting with page 1, then closing the book, opening it again on pages $2-3$, closing it, opening it on pages $4-5$, etc. But unless you're somewhat eccentric, you probably prefer to just turn the pages in between.

Iteration by pointers. Arrays offer natural and efficient iteration through *pointers*. Pointer values can be thought of as actual addresses, and they allow operations like "adding s" in order to go to the next element in the array. Here is how we could equivalently write the iteration in lines 12–13 of Program 14 with pointers.

```
bool* begin = crossed_out;        // pointer to first element
bool* end = crossed_out + 1000; // past-the-end pointer
// in the loop, pointer p successively points to all elements
for (bool* p = begin; p != end; ++p)
  *p = false; // *p is the element pointed to by p
```

Admittedly, this looks more complicated at first sight than the random access version, but we'll explain what's going on in detail in the next sections. In terms of Figure 7, we have replaced iteration by index with iteration by address.

2.6.6 Pointer types and functionality

For any type T the corresponding *pointer type* is

$T*$

We call T the underlying type of $T*$. An expression of type $T*$ is called a *pointer* (to T).

The value of a pointer to T is the address of an object of type T. We call this the object *pointed to* by the pointer.

We can visualize a pointer p as an arrow pointing to a cell in the computer's main memory—the cell where the object pointed to by p starts, see Figure 8.

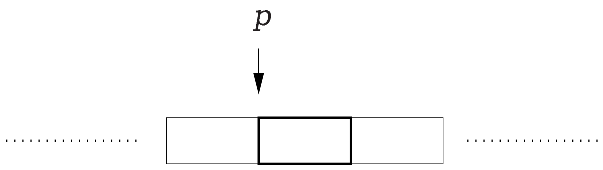


Figure 8: *A pointer to T represents the address of an object of type T in the computer's main memory.*

Initialization, the assignment operator =, and the comparison operators == and != are defined for any pointer type *T**. The latter simply test whether the addresses in question are the same or not.

Initialization and assignment copy the value (as usual), which in this case means to copy an address; thus, if `j` points to some object, the assignment `i = j` has the effect that `i` now also points to this object. The object itself is not copied. We remark that pointer initialization and assignment require the types of both operands to be exactly the same—implicit conversions don't work. If you think about it, this is clear. Imagine that the variable `i` is of type `int*`, and that you could write

```
double* j = i
```

Since `double` objects usually require more memory cells than `int` objects, `j` would now be a pointer to a `double` object that includes memory cells originally not belonging to `i`. This can hardly be called a "conversion". In fact, since we only copy an address, there cannot be any physical conversion of the stored value, even if the memory requirements of the two types happen to be the same.

The address operator. We can obtain a pointer to any given object by applying the unary *address operator* to any lvalue that refers to the object. If the lvalue is of type `T`, then the result is an rvalue of type `T*`. The syntax of an address operator call is

&lvalue

In the following code fragment we use the address operator to initialize a variable `iptr` of type `int*` with the address of an object of type `int` named `i`.

```
int i = 5;
int* iptr = &i; // iptr initialized with the address of i
```

The dereference operator. From a pointer, we can get back to the object pointed to through *dereferencing* or *indirection*. The unary *dereference operator* `*` applied to an rvalue of pointer type yields an lvalue referring to the object pointed to. If the rvalue is of type *T**, then the result is of type *T*. The syntax of a dereference operator call is

∗rvalue

Following up on our previous code fragment, we can therefore write

```
int i = 5;
int* iptr = &i; // iptr initialized with the address of i
int j = *iptr;  // j == 5
```

The naming scheme of pointer types is motivated by the dereference operator. The declaration

T∗ p

can also be read (and in fact legally be written; we don't do this, though) as

T ∗p

The second version implicitly defines the type of p by saying that **p* is of type *T*. This is the same kind of implicit definition that we already know from array declarations.

Figure 9 illustrates address and dereference operator.

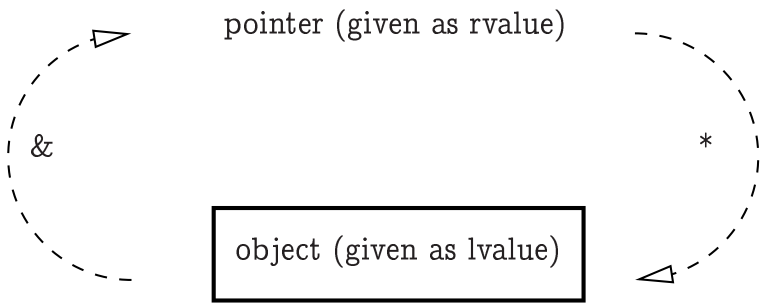


Figure 9: *The address operator (left) and its inverse, the dereference operator (right)*

The null pointer. For any pointer type there is a value distinguishable from any other pointer value. This value is called the *null pointer value*. The integer value 0 can be converted to any pointer type. The value after conversion is the null pointer value. In the declaration `int* iptr = 0`, for example, the variable `iptr` gets initialized with the null pointer value. We also say that `iptr` is a *null pointer*. The null pointer value must not be dereferenced, since it does not correspond to any existing address.

Using the null pointer value is the safe way of indicating that there is no object (yet) to point to. The alternative of leaving the pointer uninitialized is bad: there is no way of testing whether a pointer that is not a null pointer holds the address of a legitimate object, or whether it holds some "random" address resulting from leaving the pointer uninitialized.

In the latter case, dereferencing the pointer usually crashes the program. Consider this code:

```
int* iptr;      // uninitialized pointer
int j = *iptr; // trouble!
```

After its declaration, the pointer iptr has undefined value, which in practice means that it may correspond to an arbitrary address in memory; dereferencing it means to access the memory content at this address. In general, this address will not belong to the part of memory to which the program has access; the operating system will then deny access to it and terminate the program with a *segmentation fault*.

2.6.7 Array-to-pointer conversion

Any array of type $T[\mathtt{n}]$ can implicitly be converted to type $T*$. The resulting value is the address of the first element of the array. For example, we can write

```
int a[5];
int* begin = a; // begin points to a[0]
```

The declaration

```
int* begin = &a[0]; // address of the first element
```

is equivalent as far as the resulting value of begin is concerned, but there is a subtle difference: the latter declaration evaluates a[0], while the former does not.

The pointer-style replacement code for the loop in lines 12–12 of Program 14 that we have presented at the end of Section 2.6.5 makes use of array-to-pointer conversion in the first line:

```
bool* begin = crossed_out;       // pointer to first element
```

The array-to-pointer conversion is purely conceptual; on the machine side, nothing happens. For this, we recall from our earlier discussion in Section 2.6.4 that in C and therefore also in C++, an array "is" simply the address of its first element.

Array-to-pointer conversion automatically takes place when an array appears in an expression.[22] Bjarne Stroustrup, the designer of C++, illustrates this by saying that *the name of an array converts to a pointer to its first element at the slightest provocation*. In still other words, there are no operations on arrays: everything that we conceptually do with an array is in reality done with a pointer; this in particular applies to the random access operation, see the paragraph called "Pointer subscripting, or the truth about random access" in the next section.

[22]A notable exception is the case where an array appears as the left operand of an assignment. This does not trigger array-to-pointer conversion but an error message saying that arrays can't be assigned to.

2.6.8 Pointer arithmetic

In order to understand why the code fragment

```
bool* begin = crossed_out;      // pointer to first element
bool* end = crossed_out + 1000; // past-the-end pointer
// in the loop, pointer p successively points to all elements
for (bool* p = begin; p != end; ++p)
  *p = false; // *p is the element pointed to by p
```

indeed sets all elements of the array `crossed_out` to *false*, we have to understand *pointer arithmetic*, the art of computing with addresses. We deliberately call this an "art", since pointer arithmetic comes with a lot of pitfalls, but without a safety net. On the other hand, the authors feel that there is also a certain beauty in the minimalism of pointer arithmetic. It's like driving an oldtimer: it's loud, it's difficult to steer, seats are uncomfortable, and there's no heating. But the heck with it! The oldtimer looks so much better than a modern car. Nevertheless, after driving the oldtimer for a while, it will probably turn out that beauty is not enough, and that safety and usability are more important factors in the long run.

Adding integers to pointers. The binary addition operators +, - are defined for left operands of any pointer type *T** and right operands of any integral type. Recall that if an array is provided as the left operand, it will implicitly be converted to a pointer using array-to-pointer conversion.

For the behavior of + to be defined, there must be an array of some length n, such that the left operand *ptr* is a pointer to the element of some index $k, 0 \leq k \leq n$, in the array. The case $k = n$ is allowed and corresponds to the situation where *ptr* is a pointer one past the last element of the array (we call this a *past-the-end pointer*; note that such a pointer must not be dereferenced).

If the second operand *expr* has some value i such that $0 \leq k + i \leq n$, then

ptr + *expr*

is a pointer to the $(k + i)$-th element of the same array. Informally, we get a pointer that has been moved "i elements to the right" (which actually means to the left if i is negative). Therefore, if p is the value of ptr (an address), then the value of `ptr + expr` is the address $p + si$, assuming that any value of the underlying type occupies s memory cells. The pleasing fact is that we don't have to care about s; the operation `ptr + expr` (which knows s from the type of ptr) does this for us and offers a type-independent way of moving a pointer i elements to the right.

As before, if $k + i = n$, we get a past-the-end pointer. Values of i such that $k + i$ is not between 0 and n lead to undefined behavior.

Let us repeat the point that we have made before in connection with random access in Section 2.6.3: by default, there are absolutely no checks that the above requirements indeed hold, and it is entirely your responsibility to make sure that this is the case. Failure to do so will result in program crashes, strange behavior of the program, or (probably the worst scenario) seemingly normal behavior, but with the potential of turning into strange behavior at any time, or on any other machine.

Therefore, let us summarize the requirements once more:

- *ptr* must point to the element of index k in some array of length n, where $0 \leq k \leq n$, and
- *expr* must have some value i such that $0 \leq k + i \leq n$.

Binary subtraction is similar. If *expr* has value i such that $0 \leq k - i \leq n$, then

ptr − *expr*

yields a pointer to the array element of index $k - i$.

The assignment versions += and -= of the two operators can be used with left operands of pointer type as well, with the usual meaning. Similarly, the unary increment and decrement operators ++ and -- are available for pointers. Since precedences and associativities are tied to the operator symbols, they are as in Table 1 on page 48.

Now we can understand the second line of the above code fragment:

```
bool* end = crossed_out + 1000; // pointer after last element
```

First, the array `crossed_out` is converted to a pointer to its first element (the one of index 0). Since the array has $1,000$ elements, adding the integer $1,000$ yields a past-the-end pointer `end` for the array. The subsequent loop

```
for (bool* p = begin; p != end; ++p)
  *p = false; // *p is the element pointed to by p
```

is clear now as well: starting with a pointer `p` to the first element (`p = begin`), the element pointed to is set to *false* (`*p = false`). Then we increment `p` so that it points to the next element (`++p`). We repeat this as long as `p` is different from the past-the-end pointer named `end`.

Pointer comparison. We have already discussed the relational operators == and != that simply test whether the two pointers in question point to the same object. But we can also compare two pointers using the operators <, <=, >, and >=. Again, precedences and associativities of all relational operators are as in Table 2 on page 69.

For the result to be specified, there must be an array of some length n, such that the left operand *ptr1* is a pointer to the element of some index $k_1, 0 \leq k_1 \leq n$ in the array, and the second operand *ptr2* is a pointer to the element of some index $k_2, 0 \leq k_2 \leq n$ in the same array. Again, $k_1 = n$ and $k_2 = n$ are allowed and correspond to the past-the-end case.

Given this, the result of the pointer comparison is determined by the integer comparison of k_1 and k_2. In other words (and quite intuitively), the pointer to the element that comes first in the array is the smaller one.

In our code fragment, the comparison `p != end` could equivalently be replaced by the expression `p < end` which yields *true* as long as `p` points to an actual array element, equivalently as long as `p` is not a past-the-end pointer.

Comparing two pointers that do not meet the above requirements leads to unspecified results in the four operators <, <=, >, and >=.

Pointer subtraction. There is one more arithmetic operation on pointers. Assume that *ptr1* is a pointer to the element of some index $k_1, 0 \leq k_1 \leq n$ in some array of length n, and the second operand *ptr2* is a pointer to the element of some index $k_2, 0 \leq k_2 \leq n$ in the same array (past-the-end pointers allowed). Then the result of the *pointer subtraction*

$$\mathit{ptr1} - \mathit{ptr2}$$

is the integer $k_1 - k_2$. Thus, pointer subtraction tells us "how far apart" the two array elements are. The behavior of pointer subtraction is undefined if *ptr1* and *ptr2* are not pointers to elements in (or past-the-end pointers of) the same array.

Pointer subtraction (which employs the binary subtraction operator, see Table 1 on page 48 for its specifics) does not occur in the code fragment from the beginning of this section. A typical use is to determine the number of elements in an array that is given by a pointer to its first element and a past-the-end pointer.

Pointer subscripting, or the truth about random access. In reality, the subscript operator [] as introduced in Section 2.6.3 does not operate on arrays, but on pointers. Invoking this operator on an array constructs an expression and therefore triggers an array-to-pointer conversion.

Given a pointer *ptr* and an expression *expr* of integral type, the expression

$$\mathit{ptr}[\mathit{expr}]$$

is equivalent (also in its requirements on *ptr* and *expr*) to

$$*(\mathit{ptr} + \mathit{expr})$$

If *expr* has value i, the latter expression yields the array element i places to the right of the one pointed to by *ptr*. In particular, if *ptr* results from an array-to-pointer conversion, this agrees with the semantics of random access for arrays as introduced in Section 2.6.3.

Table 4 summarizes the new pointer-specific binary operators.

Description	Operator	Arity	Prec.	Assoc.
subscript	[]	2	17	left
dereference	*	1	16	right
address	&	1	16	right

Table 4: *Precedences and associativities of pointer operators. The subscript operator expects rvalues as operands and returns an lvalue. The dereference operator expects an rvalue and returns an lvalue, while the address operator expects an lvalue and returns an rvalue.*

What have we gained with pointers? So far it seems that the only use of pointers is to make iteration through an array a little more efficient than iteration by index. But unless we are in the realm of extremely time-critical loops, the savings are marginal. For the sake of readability, we therefore often still use iteration by index. So what is the real justification for the pointer concept?

There are actually two justifications, and one of them will be discussed right away in the next section: pointers are indispensable for getting "practical" arrays with length not known at compile time.

The second justification is not yet around the corner, so we will only briefly touch it here. Arrays are by far not the only containers for sets of data. When we implement data processing algorithms, we should therefore make sure that they work *not* only for arrays.

For example, finding a container element with a given property (movie theater that plays your favorite movie) should be possible for *any* containers that offers the functionality of iterating over its elements. The only uniformity we need is in the iteration process itself.

Any data-processing algorithm of the C++ standard library (we will see some of them later) works in this way: it expects the underlying container to offer *iterators* conforming to some well-defined iterator concept. The specifics of the container itself are irrelevant for the algorithm.

Here is where pointers come in: they are the iterators offered by arrays. Therefore, even if we don't use pointers in our own code, we have to know about them in order to be able to apply standard library algorithms to arrays.

2.6.9 Dynamic memory allocation

Let us go back to Program 14 now. Its main drawback is that the number n is hardwired as $1,000$ in this program, just because the length of an array has to be known at compile time.

At least in this respect, arrays are nothing special, though. All types that we have met earlier (`int`, `unsigned int`, and `bool`) have the property that a single object of the type occupies a fixed amount of memory known to the compiler (for example, 32 bits for an int object on many platforms). With arrays, an obvious need arises to circumvent this restriction.

In C++, arrays whose length is determined at runtime can be obtained through *dynamic memory allocation*. Through such an allocation, we create an object with *dynamic* storage duration.

Objects that we have seen so far were all tied to variables, in which case memory gets assigned to them (and is freed again) at predetermined points during program execution (automatic and static storage duration, Section 2.4.3). Objects of dynamic storage duration are not tied to variables, and they may "start to live" (get memory assigned to them) and "die" (get their memory freed) at *any* point during program execution. The programmer can determine these points via new and delete expressions.

The program has some (typically quite large) region of the computer's main memory available to store dynamically allocated objects. This region is called the *heap*. It is initially unused, but when an object is dynamically allocated, it is being stored on the heap, so that the memory actually used by the program grows.

Here is how this works for Eratosthenes' Sieve. Remember that we want the list of prime numbers between 2 and $n-1$. The following variant reads the number n from standard input and dynamically allocates an array of length n. The remainder of the program is as before, except that we explicitly have to free the dynamically allocated storage in the end.

```
// Program: eratosthenes2.C
// Calculate prime numbers in {2,...,n-1} using
// Eratosthenes' sieve.

#include <iostream>

int main()
{
  // input
  std::cout << "Compute prime numbers in {2,...,n-1} for n =? ";
  unsigned int n;
  std::cin >> n;

  // definition and initialization: provides us with
  // Booleans crossed_out[0],..., crossed_out[n-1]
  bool* crossed_out = new bool[n];        // dynamic allocation
  for (unsigned int i = 0; i < n; ++i)
    crossed_out[i] = false;

  // computation and output
  std::cout << "Prime numbers in {2,...," << n-1 << "}:\n";
  for (unsigned int i = 2; i < n; ++i)
    if (!crossed_out[i]) {
```

```
      // i is prime
      std::cout << i << " ";
      // cross out all proper multiples of i
      for (unsigned int m = 2*i; m < n; m += i)
        crossed_out[m] = true;
    }
  std::cout << "\n";

  delete[] crossed_out;                    // free dynamic memory

  return 0;
}
```

Program 15: *progs/eratosthenes2.C*

Note that the variable crossed_out is now a pointer rather than an array; after the new declaration, it points to the first element of a dynamically allocated array of length n.

The new expression. For any type T, a new expression can come in any of the following three variants.

new T
new $T(...)$
new $T[expr]$

In all cases, the expression returns an rvalue of type $T*$. Its value is the address of an object of type T that has been dynamically allocated on the heap. The object itself is anonymous, but we usually store the resulting address under a variable name. In Program 15, we call it crossed_out.

In the first and second variant, the effect of the new expression is to dynamically allocate a *single* object of type T on the heap. Variant 1 leaves the object uninitialized if T is a fundamental type, while variant 2 initializes the new object with whatever appears in parentheses. For example, the following declarations initialize the variables i and j, both of type int*, with the addresses of two new objects of type int.

```
int* i = new int;     // *i is undefined
int* j = new int(6);  // *j is 6
```

Right now, if we wanted two such objects of type int, we'd rather use variables with automatic storage duration and write

```
int i;                // i is undefined
int j = 6;            // j is 6
```

More interesting for us is the third variant. If *expr* has integer value $n \geq 0$, the effect of the new expression is to dynamically allocate an array of length n with underlying type T on the heap. The return value is the address of the first element. This is what we see in line 16 of Program 15.

As usual, the n array elements remain uninitialized if T is a fundamental type.

The delete expression. Dynamically allocated memory that is no longer needed should be freed. In C++, the programmer decides at which point this is the case. [23] Dynamic storage duration implies that dynamically allocated objects live until the program terminates, unless they are explicitly freed. Dynamically allocated memory is more flexible than static memory, but in return it also involves some administrative effort.

The delete expressions take care of freeing memory. They come in two variants.

```
delete expr
delete[] expr
```

In both variants, *expr* may be a null pointer, in which case the delete expression has no effect.

Otherwise, in the first variant, *expr* must be a pointer to a single object that has previously been dynamically allocated with the first or second variant of the new expression. The effect is to make the corresponding memory available again for subsequent dynamic allocations on the heap.

For example, at a point in the program where the two int objects dynamically allocated through

```
int* i = new int;      // *i is undefined
int* j = new int(6);   // *j is 6
```

are no longer needed, we would write

```
delete j;
delete i;
```

The order of deletion does not matter here, but many programmers consider it logical to delete pointers in the inverse order of dynamic allocation: If you need to undo two steps, you first undo the second step.

In the second variant of the delete expression, *expr* must be a pointer to the first element of an array that has previously been dynamically allocated with the third variant of the new expression. The whole memory occupied by the array is put back on the heap for reuse.[24] This happens in line 32 of Program 15.

If the plain delete is applied to a non-null pointer that does not point to a dynamically allocated single object, the behavior is undefined. The same is true if one tries

[23] There are programming languages (Java, for example) that automatically detect and free unused memory on the heap. This automatic process is called *garbage collection*. It is generally more user-friendly than the manual deletion process in C++, but requires a more sophisticated implementation. In any case, you are free to implement garbage collection also in C++.

[24] This implies that the length of a dynamically allocated array *is* actually stored somewhere with the heap; still, we can't access this length from the program.

to `delete[]` an array where there is only a single object. As always with pointers, the C++ language does not offer any means of detecting such errors.

Memory leaks. Although all memory allocated by a program is automatically freed when the program terminates normally, it is very bad practice to rely on this fact for freeing dynamically allocated memory. If a program does not explicitly free all dynamically allocated memory it is said to have a *memory leak*. Such leaks are often a sign of bad coding. They usually have no immediate consequences, but without freeing unused storage, a program running for a long time (think of operating system routines) may at some point simply exhaust the available heap storage.

Therefore, we have the following guideline.

> **Dynamic Storage Guideline:**
> new and delete expressions should always come in matching pairs.

2.6.10 Arrays of characters

Sequences of characters enclosed in double quotes like in

```
std::cout << "Prime numbers in {2,...,999}:\n";
```

are called *string literals*.[25]

So far we have used string literals only within output expressions, but we can work with them in other contexts as well. Most notably, a string literal can be used to initialize an array of *characters*. Characters are the building blocks of text as we know it. In C++, they are modeled by the fundamental type `char` that we briefly discuss next.

The type char. The fundamental type `char` represents characters. Characters include the *letters* a through z (along with their capital versions A through Z), the *digits* 0 through 9, as well as numerous other *special characters* like % or \$. The line

```
char c = 'a';
```

defines a variable `c` of type `char` and value `'a'`, representing the letter a. The expression `'a'` is a literal of type `char`. The quotes around the actual character symbol are necessary in order to distinguish the literal `'a'` from the identifier `a`.

Formally, the type `char` is an integral type: it has the same operators as the types `int` or `unsigned int`, and the C++ standard even postulates a promotion from `char` to `int` or `unsigned int`. It is *not* specified, though, to which integer the character `'a'`, say, will be promoted. Under the widely used `ASCII` code (**A**merican **S**tandard **C**ode for **I**nformation **I**nterchange), it is the integer 97.

[25] Unlike all other literals, string literals are lvalues, but the effect of trying to modify them is undefined; luckily, we won't need these "interesting" facts.

This setting may not seem very useful, and indeed it makes little sense to divide one character by another. On the other hand, we can for example print the alphabet through one simple loop (assuming ASCII encoding). Execution of the for-loop

```
for (char c = 'a'; c <= 'z'; ++c)
  std::cout << c;
```

writes the character sequence

```
abcdefghijklmnopqrstuvwxyz
```

to standard output. Given this, you may think that the line

```
std::cout << 'a' + 1;
```

prints 'b', but it doesn't. Since the operands of the composite expression 'a'+1 are of different types, the left operand of type char will automatically be promoted to the more general type int of the right operand. Therefore, the type of the expression 'a'+1 is int, and its value is 98 (assuming ASCII encoding); and that's what gets printed. If you want 'b' to be printed, you must use the explicit conversion char('a'+1).

The category of special characters also includes *control characters* that *do* something when printed. These are written with a leading backslash, and the most important control character for us is '\n', which causes a line break.

On most platforms, a char value occupies 8 bits of memory; whether the value range correspond to the set of integers $\{-128, \ldots, 127\}$ (the signed case) or the set $\{0, \ldots, 255\}$ (the unsigned case) is implementation defined. Since all ASCII characters have integer values in $\{0, \ldots, 127\}$, they can be represented in both cases.

From characters to text. A text is simply a sequence of characters and can be modeled in C++ through an array with underlying type char. For example, the declaration

```
char text[] = {'b', 'o', 'o', 'l'}
```

defines an array of length 4 that represents the text *bool*.

Alternatively (and more conveniently), we can write

```
char text[] = "bool"
```

This, however, is *not* equivalent to the former declaration. When an array of characters is initialized with a string literal, the terminating *zero character* '\0' (of integer value 0) is automatically appended to the array. This character does not correspond to any printable character. After the latter declaration, the array text therefore has length 5. The first four elements are 'b', 'o', 'o', and 'l', and the fifth element is the zero character '\0'.

We call such an array *zero-terminated*. Unlike normal arrays, zero-terminated arrays "know" their length. To get this length, we simply have to iterate over the array and count the number of elements before the terminating '\0'.

Here is an application of (arrays of) characters. *String matching* is the problem of finding the first or all occurrences of a given search string (usually short) in a given text (usually long).

The obvious solution is the following: assuming that the search string has length m, we compare it characterwise with the elements $1, 2, ..., m$ of the text. If a mismatch is found for some element, we stop and next compare the search string with the elements $2, 3, \ldots, m+1$ of the text, and so on. Sets of m consecutive elements $i, i+1, \ldots, i+m-1$ in the text are called a *window*.

This algorithm is fast as long as the search string is short, but it may become inefficient for long search strings (see Exercise 72). There is a more sophisticated algorithm (the *Knuth-Morris-Pratt algorithm*) that is always fast.

The following Program 16 implements the obvious algorithm. It maintains two arrays of characters, one for the search string, and one for the current window. We impose a cyclic order on the window (the first element directly follows the last one); this makes it easy to shift the window one place, by simply replacing element i of the text with element $i + m$ (and at the same time advancing the logical first position of the window by one).

```
// Program: string_matching.C
// find the first occurrence of a fixed string within the
// input text, and output the text so far

#include<iostream>

int main ()
{
  // search string
  char s[] = "bool";

  // determine search string length m
  unsigned int m = 0;
  for (char* p = s; *p != '\0'; ++p) ++m;

  // cyclic text window of size m
  char* t = new char[m];

  unsigned int w = 0; // number of characters read so far
  unsigned int i = 0; // index where t logically starts

  // find pattern in the text being read from std::cin
  std::cin >> std::noskipws; // don't skip whitespaces!

  for (unsigned int j = 0; j < m;)
    // compare search string with window at j-th element
    if (w < m || s[j] != t[(i+j)%m])
      // input text still too short, or mismatch:
      // advance window by replacing first character
```

```
      if (std::cin >> t[i]) {
        std::cout << t[i];
        ++w;          // one more character read
        j = 0;        // restart with first characters
        i = (i+1)%m;  // of string and window
      } else break;   // no more characters in the input
    else ++j; // match: go to next character

  std::cout << "\n";
  delete[] t;
  return 0;
}
```

Program 16: *progs/string_matching.C*

When we apply the program to the text of the file `eratosthenes.C`, the program outputs Program 14 up to the first occurrence of the string `"bool"`:

```
// Program: eratosthenes.C
// Calculate prime numbers in {2,...,999} using
// Eratosthenes' sieve.

#include <iostream>

int main()
{
  // definition and initialization: provides us with
  // Booleans crossed_out[0],..., crossed_out[999]
  bool
```

A few comments need to be made with respect to the handling of standard input here. The program reads the text character by character from `std::cin`, until this stream becomes "empty". To test this, we use the fact that stream values can implicitly be converted to `bool`, with the result being *true* as long as there was no attempt at reading past the end of the stream. Since the value of `std::cin >> t[i]` is the stream *after* removal of one character, the conversion to `bool` exactly tells us whether there still was a character in the stream, or not.

Most conveniently, the program is run by redirecting standard input to a file containing the text. In this case, the stream `std::cin` will become empty exactly at the end of the file. The line

```
std::cin >> std::noskipws; // don't skip whitespaces!
```

is necessary to tell the stream that *whitespaces* (blanks, newlines, etc.) should not be ignored (by default, they are). This allows us to search for strings that contain whitespaces, and it allows us to output the text (up to the first occurrence of the search string) in its original layout.

2.6.11 Multidimensional arrays

In C++, we can have arrays of arrays. For example, the declaration

```
int a[2][3]
```

declares a to be an array of length 2 whose elements are arrays of length 3 with underlying type `int`. We also say that a is a *multidimensional array* (in this case of dimensions 2 and 3). The type of a is "`int[2][3]`", and the underlying type is `int[3]`. In general, the declaration

T a[*expr1*]...[*exprk*]

defines an array a of length n_1 (value of *expr1*) whose elements are arrays of length n_2 (value of *expr2*) whose elements are... you get the picture. The values $n_1, \ldots, n_k$ are called the *dimensions* of the array, and the expressions *expr1*,...,*exprk* must be constant expressions of integral type and positive value.

Random access in multidimensional arrays works as expected: `a[i]` is the element of index i, and this element is an array itself. Consequently, `a[i][j]` is the element of index j in the array `a[i]`, and so on.

Although we usually think of multidimensional arrays as tables or matrices, the memory layout is "flat" like for one-dimensional arrays. For example, the twodimensional array declared through `int a[2][3]` occupies a contiguous part of the memory, with space for $6 = 2 \times 3$ objects of type `int`, see Figure 10.

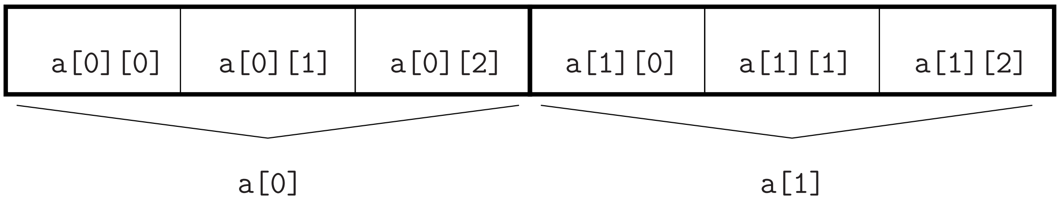


Figure 10: *Memory layout of a twodimensional array*

Multidimensional arrays can be initialized in a way similar to onedimensional arrays; the value for the *first* (and *only* the first) dimension may be omitted:

```
int a[][3] = { {2,4,6}, {1,3,5} };
```

This defines an array of type `int[2][3]` where `{2,4,6}` is used to initialize the element `a[0]`, and `{1,3,5}` is used for `a[1]`.

Dynamic allocation of multidimensional arrays. The required dimensions of a multidimensional array may not be known at compile time in which case dynamic allocation is called for. Let us start with the case where all dimensions but the first are known at compile time. If *expr* has value $n \geq 0$, a pointer to a dynamically allocated array of length n with underlying type $T[n_2]...[n_k]$ is obtained from a new expression

```
new T[expr][expr2]...[exprk]
```

where *expri* has value n_i, $i = 1, \ldots, k$. All dimensions but the first must be constant expressions. If you think about it for a minute, this is not surprising. For example, in order to generate machine language code for random access operations on the dynamically allocated array, the compiler must know how many memory cells a single element of the underlying type $T[n_2]...[n_k]$ occupies (see Section 2.6.3). But this is only possible if the values $n_2, \ldots, n_k$ are known at compile time.

Pointers to arrays. If we want to use the above new expression to initialize a pointer variable (with the address of the first element of the multidimensional array), we need the type "pointer to $T[n_2]...[n_k]$". As you may suspect, we informally call this type "$T[n_2]...[n_k]$*", but we can't write it like that in C++, since $T[n_2]...[n_k]$ is not a type name. Again, we have to resort to an implicit definition of the desired pointer variable p, as in the following code fragment.

```
int n = 2;
int (*p)[3] = new int[n][3];  // type of *p: int[3] <=> p: int[3]*
```

The parentheses are necessary here, since `int *p[3]` (which is the same as `int* p[3]`) declares p to be an array of pointers to int (see also next paragraph). C++ syntax is bittersweet.

Arrays of pointers. If you're asking for a multidimensional array with non-constant dimensions among $n_2, \ldots, n_k$, the official answer is: there is none. But under the counter, you can buy a very good imitation.

One first solution that suggests itself when you reconsider the flat memory layout of multidimensional arrays is this: you dynamically allocate a onedimensional array of length $n = n_1 \times n_2 \times \cdots \times n_k$ and artificially partition it into subarrays by doing some juggling with indices.

Let us discuss the twodimensional case only to avoid lengthy formulae. A twodimensional array with dimensions n and m can be simulated by a onedimensional array of length nm. The element with logical indices $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1, \ldots, m-1\}$ appears at index $mi + j$ in the onedimensional array. Vice versa, the element of index ℓ in the onedimensional array has logical indices $i = \ell \operatorname{div} m$ and $j = \ell \bmod m$. This works because the function

$$(i, j) \mapsto mi + j$$

bijectively maps the set of logical indices (i, j) to the set of numbers $\{0, 1, \ldots, nm-1\}$. Intuitively, this mapping flattens the imaginary table of n rows and m columns by simply putting one row after another. As you can see from Figure 10, this is exactly what the compiler is implicitly doing for multidimensional arrays with *constant* dimensions $n_1, \ldots, n_k$.

Doing it explicitly for non-constant dimensions is only a workaround, though, since we lose the intuitive notation a[i][j]; moreover, this workaround becomes even more cumbersome with higherdimensional arrays.

A better solution that keeps the notation a[i][j] and that smoothly extends to higher dimensions is the following (again, we only discuss the case of a twodimensional array with dimensions n and m): you first dynamically allocate one array of n *pointers*, and then you let every single pointer point to the first element of an individual, dynamically allocated array of length m. The following code fragment demonstrates this.

```
// a points to the first element of an array of n pointers to int
int** a = new int*[n];
for (int i = 0; i < n; ++i)
  // a[i] points to the first element of an array of m int's
  a[i] = new int[m];
```

The type int** is "pointer to pointer to int". a[i] is therefore a pointer to int (see the paragraph on pointer subscripting in Section 2.6.8), and a[i][j] is an lvalue of type int, just like in a "regular" twodimensional array.

The memory layout is different, though: Figure 10 is replaced by Figure 11. This means, the twodimensional array is patched up from a set of n onedimensional arrays, but these n arrays are not necessarily consecutively arranged in memory. In fact, the n arrays may even have different lengths. This is useful for example when you want to store a lower-triangular matrix; in this case, it suffices if the row of index i has length $i + 1$.

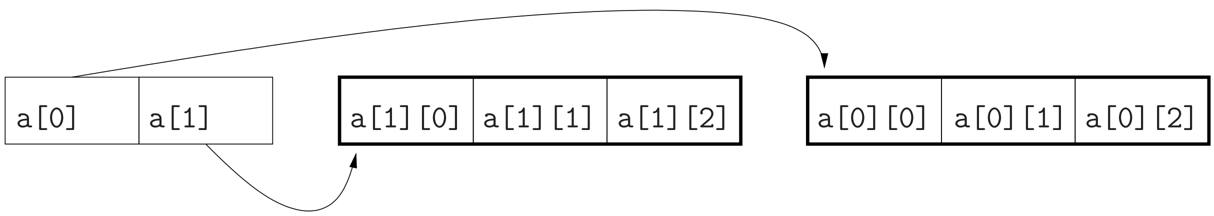


Figure 11: *Memory layout of a twodimensional array realized by an array of pointers*

Computing shortest paths. Let us conclude this section with an interesting application of (multidimensional) arrays. Imagine a rectangular factory floor, subdivided into square cells. Some of the cells are blocked with obstacles (these could for example be machines or cupboards, but let us abstractly call them "walls"). A robot is initially located at some cell S (the source), and the goal is to move the robot to some other cell T (the target). At any time, the robot can make one step from its current cell to any of the four adjacent cells, but for obvious reasons it may only use cells that are empty.

Given this setup, we want to find a shortest possible robot path from S to T (or find out that no such path exists). Here, the length of a robot path is the number of steps taken by the robot during its motion from S to T (the initial cell S does not count; in particular, it takes 0 steps to reach S from S). Figure 12 (left) shows an example with 8×12 cells.

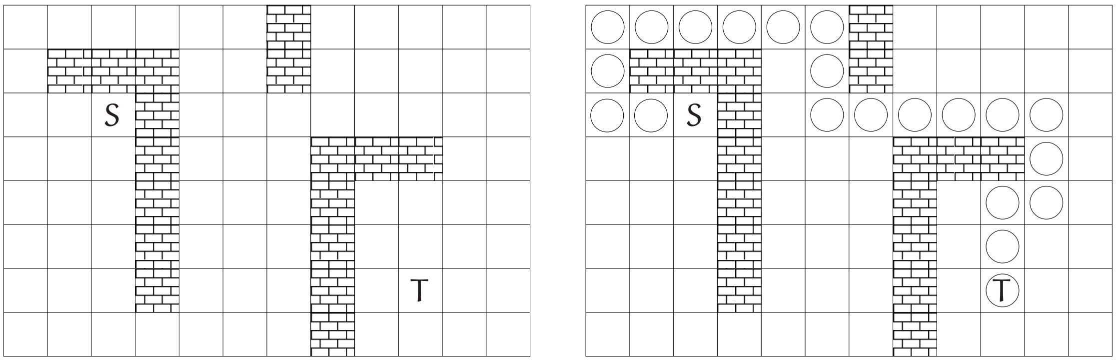


Figure 12: *Left: What is a shortest robot path from S to T? Right: This one!*

In this example, a little thinking reveals that there are essentially two different possibilities for the robot to reach T: it can pass below the component of walls adjacent to S, or above. It turns out that passing above is faster, and a resulting shortest path (of length 21) is depicted in Figure 12 (right). Note that in general there is not a unique shortest path. In our example, the final right turn of the path could also have been made one or two cells further down.

We want to write a program that finds a shortest robot path, given the dimensions n (number of rows) and m (number of columns) of the factory floor, the coordinates of source and target, and the walls. How can this be done? Before reading further, we encourage you to think about this problem for a while. Please note that the *brute-force approach* of trying all possible paths and selecting the shortest one is not an option, since the number of such paths is simply too large already for moderate floor dimensions. (Besides, how do you even generate all these paths?)

Here is an approach based on *dynamic programming*. This general technique is applicable to problems whose solutions can quickly be obtained from the solutions to smaller subproblems of the same structure. The art in dynamic programming is to find the "right" subproblems, and this may require a more or less far-reaching generalization of the original problem.

Once we have identified suitable subproblems, we solve all of them in turn, from the smaller to the larger ones, and memorize the solutions. That way, we have all the information that we need in order to quickly compute the solution to a given subproblem from the solutions of the (already solved) smaller subproblems.

In our case, we generalize the problem as follows: for *all* empty cells C on the floor, compute the *length* of a shortest path from S to C (where the value is ∞ if no such path exists). We claim that this also solves our original problem of computing a shortest path from S to T: Assume that the length of a shortest path from S to T is $\ell < \infty$ (otherwise we know right away that there is no path at all). We also say that T is *reachable* from S in ℓ steps.

Now if $T \neq S$, there must be a cell adjacent to T that is reachable from S in $\ell - 1$ steps, and adjacent to this a cell reachable in $\ell - 2$ steps etc. Following such a chain of cells until we get to S gives us a path of length ℓ which is shortest possible.

Let us rephrase the generalized problem: we want to label any empty cell C with a nonnegative integer (possibly ∞) that indicates the length of a shortest path from S to C. Here are the subproblems to which we plan to reduce this: for a given integer $i \geq 0$, label all the cells that are reachable from S in at most i steps. For $i = nm - 1$ (actually, for some smaller value), this labels all cells that are reachable from S at all, since a shortest path will never enter any cell twice.

Here is the reduction from larger to smaller subproblems: assume that we have already solved the subproblem for $i - 1$, i.e. we have labeled all cells that are reachable from S within $i - 1$ or less steps. In order to solve the subproblem for i, we still need to label the cells that are reachable in i steps (but not less). But this is simple, since these cells are exactly the unlabeled ones adjacent to cells with label $i - 1$.

Figure 13 illustrates how the frontier of labeled cells grows in this process, for $i = 0, 1, 2, 3$.

Continuing in this fashion, we finally arrive at the situation depicted in Figure 14: all empty cells have been labeled (and are in fact reachable from S in this example). To find a shortest path from S to T, we start from T (which has label 21) and follow any path of decreasing labels $(20, 19, \ldots)$ until we finally reach S.

The shortest path program. Let's get to the C++ implementation of the above method. We represent the floor by a dynamically allocated twodimensional array `floor` with dimensions $n + 2$ and $m + 2$ and entries of type `int`. (Formally, `floor` is a pointer to the first element of an array of $n + 2$ pointers to `int`, but we still call this a twodimensional array). These dimensions leave space for extra walls surrounding the floor. Such extra walls allow us to get rid of special cases: floor cells having less than four adjacent cells. In general, an artificial data item that guards the actual data against special cases is called a *sentinel*.

The heart of the program (which appears as Program 17 below) is a loop that computes the solution to subproblem i from the solution to subproblem $i-1$, for $i = 1, 2, \ldots$. The solution to subproblem 0 is readily available: we set the `floor` entry corresponding to S to 0, and the entries corresponding to the empty cells to -1 (this is meant to indicate that the cell has not been labeled yet). Walls are always labeled with the integer -2.

In iteration i of the loop, we simply go through all the yet unlabeled cells and label exactly the ones with i that have an adjacent cell with label $i - 1$. The loop terminates as soon as no progress is made anymore, meaning that no new cell could be labeled in the current iteration. Here is the code.

```
// main loop: find and label cells reachable in i=1,2,... steps
for (int i=1;; ++i) {
```

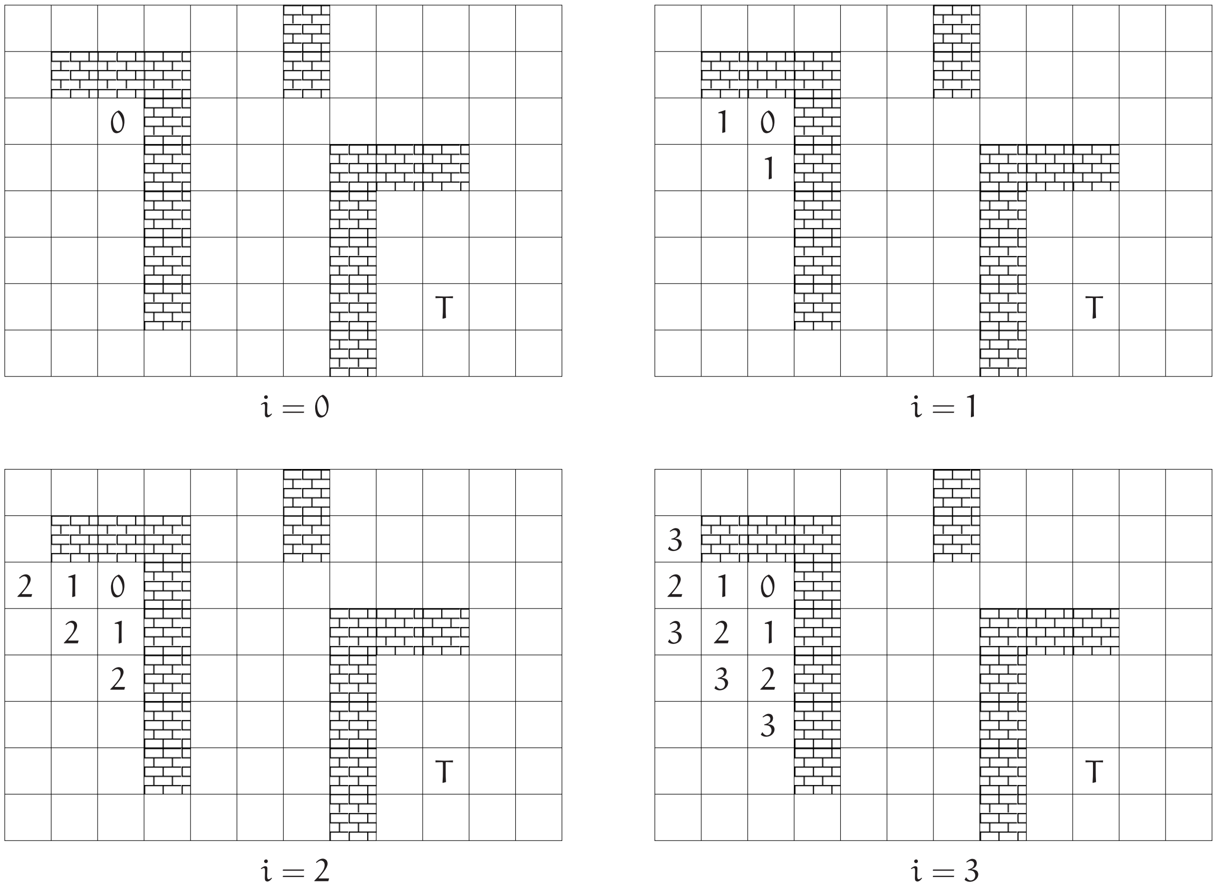
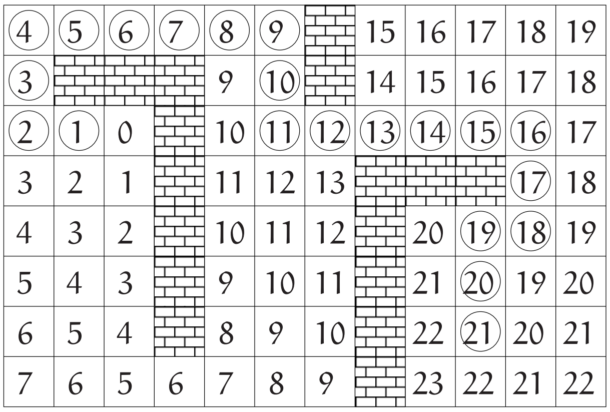


Figure 13: *The solution to subproblem* i *labels all cells* C *reachable from* S *within at most* i *steps with the length of the shortest path from* S *to* C.

```
  bool progress = false;
   for (int r=1; r<n+1; ++r)
     for (int c=1; c<m+1; ++c) {
      if (floor[r][c] != -1) continue; // wall, or labeled before
      // is any neighbor reachable in i-1 steps?
      if (floor[r-1][c] == i-1 || floor[r+1][c] == i-1 ||
          floor[r][c-1] == i-1 || floor[r][c+1] == i-1 ) {
        floor[r][c] = i; // label cell with i
        progress = true;
      }
    }
  if (!progress) break;
}
```

The other parts of the main function are more or less straightforward. Initially, we read the dimensions from standard input and do the dynamic allocation.



i = 23

Figure 14: *The solution to subproblem* i = 23 *solves the generalized problem and the original problem (a shortest path is obtained by starting from* T *and following a path of decreasing labels).*

```
// read floor dimensions
int n; std::cin >> n; // number of rows
int m; std::cin >> m; // number of columns

// dynamically allocate twodimensional array of dimensions
// (n+2) x (m+2) to hold the floor plus extra walls around
int** floor = new int*[n+2];
for (int r=0; r<n+2; ++r)
  floor[r] = new int[m+2];
```

Next, we read the floor plan from standard input. We assume that it is given rowwise as a sequence of nm characters, where 'S' and 'T' stand for source and target, 'X' represents a wall, and '-' an empty cell. The input file for our initial example from Figure 12 would then look as in Figure 15

If other characters are found in the input (or if the input prematurely becomes empty), we generate empty cells. While reading the floor plan, we put the appropriate integers into the entries of floor, and we remember the target position for later.

```
// target coordinates, set upon reading 'T'
int tr = 0;
int tc = 0;

// assign initial floor values from input:
// source:     'S'  ->     0 (source reached in 0 steps)
// target:     'T'  ->    -1 (number of steps still unknown)
// wall:       'X'  ->    -2
// empty cell: '-'  ->    -1 (number of steps still unknown)
```

```
8 12
------X-----
-XXX--X-----
--SX--------
---X---XXX--
---X---X----
---X---X----
---X---X-T--
-------X----
```

Figure 15: *Input for Program 17 corresponding to the example of Figure 12*

```
for (int r=1; r<n+1; ++r)
  for (int c=1; c<m+1; ++c) {
    char entry = '-';
    std::cin >> entry;
    if      (entry == 'S') floor[r][c] = 0;
    else if (entry == 'T') floor[tr = r][tc = c] = -1;
    else if (entry == 'X') floor[r][c] = -2;
    else if (entry == '-') floor[r][c] = -1;
  }
```

Now we add the surrounding walls as sentinels.

```
// add surrounding walls
for (int r=0; r<n+2; ++r)
  floor[r][0] = floor[r][m+1] = -2;
for (int c=0; c<m+2; ++c)
  floor[0][c] = floor[n+1][c] = -2;
```

Next comes the main loop that we have already discussed above. It labels all reachable cells, so that we obtain a labeling as in Figure 14. From this labeling, we must now extract the shortest path from S to T. As explained above, this can be done by following a chain of adjacent cells with decreasing labels. For every cell on this path (except S), we put the integer -3 into the corresponding `floor` entry; this allows us to draw the path in the subsequent output. If no path was found (or if there is no target), the body of the `while` statement in the following code fragment is (correctly) not executed at all.

```
// mark shortest path from source to target (if there is one)
int r = tr; int c = tc; // start from target
while (floor[r][c] > 0) {
  int d = floor[r][c] - 1; // distance one less
  floor[r][c] = -3; // mark cell as being on shortest path
  // go to some neighbor with distance d
  if      (floor[r-1][c] == d) --r;
```

```
  else if (floor[r+1][c] == d) ++r;
  else if (floor[r][c-1] == d) --c;
  else                         ++c; // (floor[r][c+1] == d)
}
```

Finally, the output: we map the integer entries of `floor` back to characters, where −3 becomes `'o'`, our path symbol. Inserting `'\n'` at the right places, we obtain a copy of the input floor, with the shortest path appearing in addition. We must also not forget to delete the dynamically allocated arrays in the end.

```
// print floor with shortest path
for (int r=1; r<n+1; ++r) {
  for (int c=1; c<m+1; ++c)
    if      (floor[r][c] == 0)   std::cout << 'S';
    else if (r == tr && c == tc) std::cout << 'T';
    else if (floor[r][c] == -3)  std::cout << 'o';
    else if (floor[r][c] == -2)  std::cout << 'X';
    else                         std::cout << '-';
  std::cout << "\n";
}

// delete dynamically allocated arrays
for (int r=0; r<n+2; ++r)
  delete[] floor[r];
delete[] floor;

return 0;
```

In case of our initial example, the output looks like in Figure 16. Program 17 shows the complete source code.

```
ooooooX-----
oXXX-oX-----
ooSX-oooooo-
---X---XXXo-
---X---X-oo-
---X---X-o--
---X---X-T--
-------X----
```

Figure 16: *Output of Program 17 on the input of Figure 15*

```
#include<iostream>

```

```
int main()
{
  // read floor dimensions
  int n; std::cin >> n; // number of rows
  int m; std::cin >> m; // number of columns

  // dynamically allocate twodimensional array of dimensions
  // (n+2) x (m+2) to hold the floor plus extra walls around
  int** floor = new int*[n+2];
  for (int r=0; r<n+2; ++r)
    floor[r] = new int[m+2];

  // target coordinates, set upon reading 'T'
  int tr = 0;
  int tc = 0;

  // assign initial floor values from input:
  // source:     'S'  ->     0 (source reached in 0 steps)
  // target:     'T'  ->    -1 (number of steps still unknown)
  // wall:       'X'  ->    -2
  // empty cell: '-'  ->    -1 (number of steps still unknown)
  for (int r=1; r<n+1; ++r)
    for (int c=1; c<m+1; ++c) {
      char entry = '-';
      std::cin >> entry;
      if      (entry == 'S') floor[r][c] = 0;
      else if (entry == 'T') floor[tr = r][tc = c] = -1;
      else if (entry == 'X') floor[r][c] = -2;
      else if (entry == '-') floor[r][c] = -1;
    }

  // add surrounding walls
  for (int r=0; r<n+2; ++r)
    floor[r][0] = floor[r][m+1] = -2;
  for (int c=0; c<m+2; ++c)
    floor[0][c] = floor[n+1][c] = -2;

  // main loop: find and label cells reachable in i=1,2,... steps
  for (int i=1;; ++i) {
    bool progress = false;
    for (int r=1; r<n+1; ++r)
      for (int c=1; c<m+1; ++c) {
        if (floor[r][c] != -1) continue; // wall, or labeled before
        // is any neighbor reachable in i-1 steps?
        if (floor[r-1][c] == i-1 || floor[r+1][c] == i-1 ||
```

```
            floor[r][c-1] == i-1 || floor[r][c+1] == i-1 ) {
          floor[r][c] = i; // label cell with i
          progress = true;
        }
      }
    if (!progress) break;
  }

  // mark shortest path from source to target (if there is one)
  int r = tr; int c = tc; // start from target
  while (floor[r][c] > 0) {
    int d = floor[r][c] - 1; // distance one less
    floor[r][c] = -3; // mark cell as being on shortest path
    // go to some neighbor with distance d
    if      (floor[r-1][c] == d) --r;
    else if (floor[r+1][c] == d) ++r;
    else if (floor[r][c-1] == d) --c;
    else                         ++c; // (floor[r][c+1] == d)
  }

  // print floor with shortest path
  for (int r=1; r<n+1; ++r) {
    for (int c=1; c<m+1; ++c)
      if      (floor[r][c] == 0)   std::cout << 'S';
      else if (r == tr && c == tc) std::cout << 'T';
      else if (floor[r][c] == -3)  std::cout << 'o';
      else if (floor[r][c] == -2)  std::cout << 'X';
      else                         std::cout << '-';
    std::cout << "\n";
  }

  // delete dynamically allocated arrays
  for (int r=0; r<n+2; ++r)
    delete[] floor[r];
  delete[] floor;

  return 0;
}
```

Program 17: *progs/shortest_path.C*

2.6.12 Beyond arrays and pointers

Arrays are very useful for many tasks and allow us to solve nontrivial problems like finding shortest paths in the previous section. From a theoretical point of view, arrays are in fact the only containers that we need.

On the other hand, there are two main drawbacks of arrays that we want to recapitulate here.

Arrays have fixed length. Any array, even if it is dynamically allocated, has a fixed length. In other words, we have to know *before* defining or dynamically allocating an array how many elements we need to store in it. Often, this is unrealistic. For example, in some application we might need to store a sequence of input numbers, but we don't know in advance how many numbers we will get. A typical "solution" is to dynamically allocate a very large array and just hope that the sequence fits in. The problems with this and a better (but still cumbersome) solution are outlined in Exercise 75.

A "real" solution is possible in C++ through the use of *vectors*. These are containers from the standard library that combine the classical array functionality (and its efficiency) with the possibility of growing (and shrinking) in length. Vectors can be implemented on top of arrays, and they have something similar to the mechanism outlined in Exercise 75 "built in". Vectors also largely remove the necessity of working with pointers. We will get to vectors (and their realization) later in this book.

Arrays are insecure. The usage of out-of-bound array indices is not detected in C++, and the same holds for pointers to addresses where no program object lives. With some care, you can write small programs that use arrays and pointers in a correct manner, but in complex programs, this is not easy at all. Debugging facilities of modern compilers can help, but even well-tested and frequently used large programs do not necessarily get it right. In fact, some people (let's call them attackers) are making a business of exploiting programming errors related to arrays and pointers in order to create malicious software.

Suppose that the attacker knows that some program—think of an operating system routine or a webserver—may (unintentionally) write input data beyond the bounds of an array. Due to the von-Neumann architecture, the part of the main memory being accidentally modified in this way may contain the actual program instructions. The attacker may then be able to prepare an input to the program in such a way that the program modifies itself to do whatever the attacker wants it to do. This modification runs with the same access rights as the original one, and these might be administrator rights in the worst case.

In this way, an attacker could "hijack" the computer that runs the program, and subsequently misuse it for illegal activities like sending spam, or paralyzing web servers by flooding them with requests.

For us that we are not (yet) professional programmers, the security aspect is less of a concern here. More important is that programming errors due to improper use of arrays and pointers can be very hard to find and often remain undetected until they suddenly result in strange and seemingly inexplicable behavior of the program. Also here, using vectors instead of arrays helps, since there are many potential errors related to arrays and pointers that you simply cannot make with vectors.

Why arrays, after all? Now you may ask why we have introduced arrays and pointers at all when there are more flexible and safer alternatives. Here are the three reasons.

1. Arrays and pointers are the simplest models of important standard library concepts (container and iterator).

2. Unlike vectors, arrays can be introduced without the need to discuss syntactical and semantical aspects of C++ functions and classes (that we simply don't have at our disposal at this point);

3. In order to really understand later how standard library containers and iterators are realized, it is necessary to know about arrays and pointers.

The take-home message here is this: it is important to get familiar with the *concepts* behind arrays and pointers, but it is less important to be able to actually program with arrays and pointers on a large scale.

2.6.13 Details

Constant expressions. If you want to use Eratosthenes' Sieve to compute all prime numbers smaller than $10,000$, you have to change Program 14 in several places; not counting the comments that should be updated as well, you need to replace four 1000's by 10000's. This is cumbersome and error-prone. What you want is a program that specifies the upper bound value in just one place. For this, we need a mechanism that allows us to give a name like n to a constant expression like 1000. Here is how this can be done for Eratosthenes' Sieve.

```
// Program: eratosthenes.C
// Calculate prime numbers in {2,...,n-1} using
// Eratosthenes' sieve.

#include <iostream>

int main()
{
  // define a constant n
  const unsigned int n = 1000;

  // definition and initialization: provides us with
  // Booleans crossed_out[0],..., crossed_out[n-1]
  bool crossed_out[n];
  for (unsigned int i = 0; i < n; ++i)
    crossed_out[i] = false;

  // computation and output
```

```
  std::cout << "Prime numbers in {2,...," << n-1 << "}:\n";
  for (unsigned int i = 2; i < n; ++i)
    if (!crossed_out[i]) {
      // i is prime
      std::cout << i << " ";
      // cross out all proper multiples of i
      for (unsigned int m = 2*i; m < n; m += i)
        crossed_out[m] = true;
    }
  std::cout << "\n";

  return 0;
}
```

Program 18: *progs/eratosthenes_n.C*

The keyword const in front of n's definition makes n a non-modifiable *constant* which serves as a fully-fledged constant expression. The compiler will not allow you to change the value of a constant. For example, the following leads to an error message during compilation.

```
const unsigned int n = 1000;
n = 10000; // error: can't assign to a constant
```

It is also not allowed to leave a constant uninitialized, as there is no chance to assign a value to it later:

```
const unsigned int n; // error: uninitialized constant
```

This mechanism ensures that the value of a constant is known at compile time, just like the value of the literal 1000.

Command line arguments. In Program 16 for string matching, it is not very convenient that the search string is *fixed*. We then have to recompile the program every time we want to search for another string.

A more flexible alternative is to pass the search string as a *command line argument* that we provide upon calling the program.

The main function can access such command line arguments if we provide suitable parameters. Here is how the first ten lines of Program 16 have to be changed in order to make this work.

```
// Program: string_matching2.C
// find the first occurrence of a string (provided as command
// line argument) within the input text, and output text so far

#include<iostream>

int main (int argc, char* argv[])
```

```
{
  if (argc < 2) {
    // no command line arguments (except program name)
    std::cout << "Usage: string_matching2 <string>\n";
    return 1;
  }

  // search string: second command line argument
  char* s = argv[1];
```

The values of `argc` and and `argv[]` (which is an array of pointers each of which in turn points to the first elements of a zero-terminated array of characters) are initialized by the operating system when it calls the `main` function. We will explain function parameters in detail later, here we will be satisfied with an example. Suppose that we call the program like this (assuming a Unix-type system):

```
./string_matching2 bool
```

Then `argc` (which counts the number of command line arguments) gets initialized with value 2. This count includes the program name itself (`"string_matching2"` in this case), and any additional strings provided on the command line (just the single string `"bool"` in this case). The 2 arrays `argv[0]` and `argv[1]` get initialized with the strings `"string_matching2"` and `"bool"` as described in Section 2.6.10 above. Consequently, after its definition, the pointer variable `s` in the above piece of code points to the first element of a zero-terminated array of characters that corresponds to the string `"bool"`. This gets us back to the situation in Program 16 after line 10, and the remainders of both programs are identical.

2.6.14 Goals

Dispositional. At this point, you should ...

1) know what an array is, and what random access and iteration mean in the context of arrays;
2) understand the pointer concept, and how to compute with addresses;
3) be aware that (and understand why) arrays and pointers must be used with care;
4) know that characters and arrays of characters can be used to perform basic text processing tasks;
5) know that (multidimensional) arrays of variable length can be obtained by dynamic memory allocation;

Operational. In particular, you should be able to ...

(G1) read, understand, and argue about simple programs involving arrays and pointers;

(G2) write programs that define array variables or dynamically allocate (multidimensional) arrays;

(G3) write programs that read a sequence of data into a (dynamically allocated / multidimensional) array;

(G4) write programs that perform simple data processing tasks by using random access in (multidimensional) arrays as the major tool;

(G5) within programs, iterate over a (dynamically allocated / multidimensional) array by using pointer arithmetic;

(G6) write programs that perform simple text processing tasks with arrays of characters;

2.6.15 Exercises

Exercise 65

a) What does the following program output, and why?

```
#include<iostream>

int main()
{
  int a[] = {5, 6, 2, 3, 1, 4, 0};
  int* p = a;
  do {
    std::cout << *p << " ";
    p = a + *p;
  } while (p != a);

  return 0;
}
```

b) More generally, suppose that in the previous program, a *is initialized with some sequence of* n different *numbers in* $\{0, \ldots, n-1\}$ *(we see this for* $n = 7$ *in the previous program). Prove that the program terminates in this case.*

(G1)

Exercise 66 *Assume that in some program,* a *is an array of underlying type* int *and length* n.

a) Given a variable i *of type* int *with value* $0 \leq i \leq n$, *how can you obtain a pointer* p *to the element of index* i *in* a*? (Note: if* $i = n$, *this is asking for a past-the-end pointer.)*

b) Given a pointer p *to some element in* a*, how can you obtain the index* i *of this element? (Note: if* p *is a past-the-end pointer, the index is defined as* n*.)*

Write code fragments that compute p *from* i *in a) and* i *from* p *in b).* (G1)

Exercise 67 *Let us call a natural number* k-composite *if and only if it is divisible by exactly* k *different prime numbers. For example, prime powers are 1-composite, and* $6 = 2 \cdot 3$ *as well as* $20 = 2 \cdot 2 \cdot 5$ *are 2-composite. Write a program* k_composite.C *that reads numbers* $n \geq 0$ *and* $k \geq 0$ *from the input and then outputs all* k*-composite numbers in* $\{2, \ldots, n-1\}$*. How many 7-composite numbers are there for* $n = 1{,}000{,}000$*?* (G2)(G4)

Exercise 68 *Write a program* invert.C *that inverts a* 3×3 *matrix* A *with real entries. The program should read the nine matrix entries from the input, and then output the inverse matrix* A^{-1} *(or the information that the matrix* A *is not invertible). In addition, the program should output the matrix* AA^{-1} *in order to let the user check whether the computation of the inverse was accurate (in the fully accurate case, the latter product is the identity matrix).*

Hint: *For the computation of the inverse, you can employ* Cramer's rule*. Applied to the computation of the inverse, it yields that* A^{-1}_{ij} *(the entry of* A^{-1} *in row* i *and columns* j*) is given by*

$$A^{-1}_{ij} = \frac{(-1)^{i+j}\det(A^{ji})}{\det(A)},$$

where $\det(M)$ *is the determinant of a square matrix* M*, and* A^{ij} *is the* 2×2 *matrix obtained from* A *by deleting row* j *and column* i*.*

To compute the determinant of a 3×3 *matrix, you might want to use the well-known* Sarrus' rule*.* (G2)(G3)(G4)

Exercise 69 *Write a program* read_array *that reads a sequence of* n *integers from standard input into an array. The number* n *is the first input, and then the program expects you to input another* n *values. After reading the* n *values, the program should output them in the same order. (If you can do this, you have proven that you are no longer a complete novice, according to Stroustrup.) For example, on input* `5 4 3 6 1 2` *the program should output* `4 3 6 1 2`*.* (G2)(G3)

Exercise 70 *Enhance the program* read_array.C *from Exercise 69 so that the resulting program* sort_array.C *sorts the array elements into ascending order before outputting them. Your sorting algorithm does not have to be particularly efficient, the main thing here is that it works correctly. Test your program on some larger inputs (preferably read from a file, after redirecting standard input). For example, on input* `5 4 3 6 1 2` *the program should output* `1 2 3 4 6`*.* (G2)(G3)(G4)

Exercise 71 *Enhance the program* `read_array.C` *from Exercise 69 so that the resulting program* `cycles.C` *interprets the input sequence of* n *integers as a* permutation π *of* $\{0, \ldots, n-1\}$*, and that it outputs the* cycle decomposition *of* π*.*

Some explanations are in order: a permutation π *is a bijective mapping from the set* $\{0, \ldots, n-1\}$ *to itself; therefore, the input sequence can be interpreted as the sequence of values* $\pi(0), \ldots, \pi(n-1)$ *of a permutation* π *if and only if it contains every number from* $\{0, \ldots, n-1\}$ *exactly once.*

The program `cycles.C` *should first check whether the input sequence satisfies this condition, and if not, terminate with a corresponding message. If the input indeed encodes a permutation* π*, the program should output the* cycle decomposition *of* π*. A cycle in* π *is any sequence of the form* $(\ n_1\ n_2\ \cdots\ n_k\ )$ *such that*

- $n_2 = \pi(n_1), \quad n_3 = \pi(n_2), \quad \ldots, \quad n_k = \pi(n_{k-1})$*, and* $n_1 = \pi(n_k)$*, and*
- n_1 *is the smallest element among* $n_1, \ldots, n_k$*.*

Any cycle uniquely determines the π*-values of all its elements; on the other hand, every element appears in some cycle (which might be of the trivial form* (n_1)*, meaning that* $\pi(n_1) = n_1$*). This implies that the permutation decomposes into a unique set of cycles. For example, the permutation* π *given by*

$$\pi(0) = 4, \quad \pi(1) = 2, \quad \pi(2) = 3, \quad \pi(3) = 1, \quad \pi(4) = 0$$

decomposes into the two cycles $(\ 0\ 4\ )$ *and* $(\ 1\ 2\ 3\ )$*.* (G2)(G3)(G4)

Exercise 72 *Consider the string matching algorithm of Program 16. Prove that for all* $m > 1, n \geq m$*, there exists a search string* s *of length* m *and a text* t *of length* n *on which the algorithm in Program 16 performs* $m(n-m+1)$ *comparisons between single characters.* (G1)

Exercise 73 *Consider the following program that defines and initializes a threedimensional array.*

```
#include <iostream>

int main()
{
  int a[4][2][3] =
    { // the 4 elements of a:
      { // the 2 elements of a[0]:
        {2, 4, 5}, // the three elements of a[0][0]
        {4, 6, 7}  // the three elements of a[0][1]
      },
      { // the 2 elements of a[1]:
        {1, 5, 9}, // the three elements of a[1][0]
        {4, 6, 1}  // the three elements of a[1][1]
```

```
      },
      { // the 2 elements of a[2]:
        {5, 9, 0}, // the three elements of a[2][0]
        {1, 5, 3}  // the three elements of a[2][1]
      },
      { // the 2 elements of a[3]:
        {6, 7, 7}, // the three elements of a[3][0]
        {7, 8, 5}  // the three elements of a[3][1]
      }
    };

  return 0;

}
```

Write a program `threedim_array.C` *that enhances this program by a (nested) loop that iterates over the array* `a` *and its subarrays to output all the 24* `int` *values that are stored in* `a` *and its subarrays. Do not use random access to do this but pointer arithmetic.* (G5)

Exercise 74 *Write a program* `frequencies.C` *that reads a text from standard input (like in Program 16) and outputs the frequencies of the letters in the text, where we do not distinguish between lower and upper case letters. For this exercise, you may assume that the type* `char` *implements ASCII encoding. This means that all characters have integer values in* $\{0, 1, \ldots, 127\}$*. Moreover, in ASCII, the values of the 26 upper case literals* `'A'` *up to* `'Z'` *are consecutive numbers in* $\{65, \ldots, 90\}$*; for the lower case literals* `'a'` *up to* `'z'`*, the value range is* $\{97, \ldots, 122\}$*.* (G6)

Running this on the lyrics of *Yesterday* (The Beatles) for example should yield the following output.

```
Frequencies:
a:      45 of 520
b:      5 of 520
c:      5 of 520
d:      28 of 520
e:      65 of 520
f:      4 of 520
g:      13 of 520
h:      27 of 520
i:      27 of 520
j:      0 of 520
k:      3 of 520
l:      20 of 520
m:      10 of 520
n:      30 of 520
o:      43 of 520
p:      4 of 520
q:      0 of 520
r:      19 of 520
s:      36 of 520
t:      31 of 520
u:      9 of 520
v:      6 of 520
w:      19 of 520
x:      0 of 520
y:      34 of 520
z:      0 of 520
Other: 37 of 520
```

2.6.16 Challenges

Exercise 75 *The fact that an array has fixed length is often inconvenient. For example, in Exercise 69 and in Exercise 70, the number of elements to be read into the* *array had to be provided as the first input in order for the program to be able to dynamically allocate an array of the appropriate length. But in practice, the length of the input sequence is often not known a priori.*

We would therefore like to write a program that reads a sequence of integers from standard input into an array, where the length of the sequence is not known beforehand (and not part of the input)—the program should simply read one number after another until the stream becomes empty.

One possible strategy is to dynamically allocate an array of large length, big enough to store any possible input sequence. But if the sequence is short, this is a huge waste of memory, and if the sequence is very long, the array might still not be large enough.

a) *Write a program* `read_array2.C` *that reads a sequence of integers of unknown length into an array, and then outputs the sequence. The program should satisfy the following two properties.*

   (i) *The amount of dynamically allocated memory in use by the program should at any time be* proportional to the number of sequence elements that have been read so far. *To be concrete: there must be a positive constant* a *such that no more than* ak *cells of dynamically allocated memory are in use when* k *elements have been read,* $k \geq 1$. *We refer to this property as* space efficiency. *It ensures that even very long sequences can be read (up to the applicable memory limits), but that short sequences consume only little memory.*

   (ii) *The number of assignments (of values to array elements) performed so far should at any time be proportional to the number of sequence elements that have been read so far, with the same meaning of proportionality as above. We refer to this property as* time efficiency. *It ensures that the program is only by a constant factor slower than the program* `read_array.C` *that knows the sequence length in advance.*

b) *Determine the constants of proportionality* a *for properties (i) and (ii) of your program.*

Exercise 76 *For larger floors, Program 17 can become quite inefficient, since every step* i *examines* all *cells of the floor in order to find the (possibly very few) ones that have to be labeled with* i *in that step. A better solution would be to examine only the neighbors of the cells that are already labeled with* $i-1$, *since only these are candidates for getting label* i.

Write a program `shortest_path_fast.C` *that realizes this idea, and measure the performance gain on some larger floors of your choice.*

Exercise 77 *In 1772, Leonhard Euler discovered the quadratic polynomial*

$$n^2 + n + 41$$

with the following remarkable property: if you evaluate it for $n = 0, 1, \ldots, 39$*, you always get a prime number, and moreover, all these prime numbers are different. Here is the list of all the* 40 *prime numbers generated by Euler's polynomial:*

41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601.

Here we are concerned with the question whether there are still better quadratic polynomials in the sense that they generate even more prime numbers. We say that a quadratic polynomial $an^2 + bn + c$ *has* Euler quality p *if the* p *numbers*

$$|an^2 + bn + c|, \quad n = 0, ..., p - 1$$

are different prime numbers. By taking absolute values, we therefore also allow "negative primes". As an example, let us look at the polynomial $n^2 - 10n + 2$*. For* $n = 0$*, we get* 2 *(prime), for* $n = 1$ *we obtain* -7 *(negative prime), and* $n = 2$ *gives* -14 *(no (negative) prime). The Euler quality of* $n^2 - 10n + 2$ *is therefore* 2 *but not higher.*

Here is the challenge: write a program that systematically searches for a quadratic polynomial with high Euler quality. The goal is to find a polynomial with Euler quality larger than 40*, in order to "beat"* $n^2 + n + 41$*. What is the highest Euler quality that you can find?*

For this challenge, it can be useful to read the paragraph on constant expressions in the Details section.

Exercise 78 *The* `XBM` *file format is a format for storing monochrome (black & white) images. The format is somewhat outdated, but many browsers (Internet Explorer is a notable exception) can still display images in* `XBM` *format.*

An `XBM` *image file for an image named* `test` *might look like this (taken from Wikipedia's* `XBM` *page).*

```
#define test_width 16
#define test_height 7
static char test_bits[] = {
  0x13, 0x00, 0x15, 0x00, 0x93, 0xcd, 0x55,
  0xa5, 0x93, 0xc5, 0x00, 0x80, 0x00, 0x60};
```

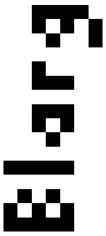
As you can guess from this, `XBM` *files are designed to be integrated into C and C++ source code which makes it easy to process them (there is no need to read in the data; simply include the file from the C++ program that needs to process the image). In our example,* `test_width` *and* `test_height` *denote the width and height of the image in pixels. Formally, these names are* macros*, but in the program they can be used like constant expressions.* `test_bits` *is an array of characters that encodes the colors of the* 16×7 *pixels in the image. Every* hexadecimal literal *of the form* $0xd_1d_2$ *encodes eight pixels, where the order is row by row. In our case,* 0x13 *and* 0x00 *encode the 16 pixels of the first row, while* 0x15 *and* 0x00 *are for the second row, etc.*

Here is how a two-digit hexadecimal literal encodes the colors of eight consecutive pixels within a row.[26] *Every hexadecimal digit* $\mathrm{d_i}$ *is from the set* $\{0, ..., 9, \mathrm{a}, \ldots, \mathrm{f}\}$ *where* a *up to* f *stand for* $10, \ldots, 15$*. The actual number encoded by a hexadecimal literal is* $16\mathrm{d}_1 + \mathrm{d}_2 \in \{0, \ldots, 255\}$*.*[27] *For example,* 0x13 *has value* $1 \cdot 15 + 3 = 19$*.*

Now, any number in $\{0, \ldots, 255\}$ *has a binary representation with 8 bits.* 19*, for example, has binary representation* 00010011*. The pixel colors are obtained by reading this backwards, and interpreting* 1 *as black and* 0 *as white. Thus, the first eight pixels in row* 1 *of the* test *image are black, black, white, white, black, white, white, white. The complete* test *image looks like this:*

Write a program xbm.C *that* #include*s an* XBM *file of your choice (you may search the web to find suitable* XBM *files), and that outputs an* XBM *file for the same image,* rotated by 90 degrees*. The program may write the resulting file to standard output. In case of the* test *image, the resulting* XBM *file and the resulting rotated image are as follows.*

```
#define rotated_width 7
#define rotated_height 16
static char rotated_bits[] = {
   0x3c, 0x54, 0x48, 0x00,
   0x04, 0x1c, 0x00, 0x1c,
   0x14, 0x08, 0x00, 0x1f,
   0x00, 0x0a, 0x15, 0x1f};
```



Note that we now have 16 instead of 14 hexadecimal literals. This is due to the fact that each of the 16 rows needs one literal for its 7 pixels, where the leading bits of the binary representations are being ignored.

You may extend your program to perform other kinds of image processing tasks of your choice. Examples include color inversion (replace black with white, and vice versa), computing a mirror image, scaling the image (so that it occupies less or more pixels), etc.

[26] If the width is not a multiple of 8, the superfluous color values from the last hexadecimal literal of each row are being ignored.

[27] If the type char has value range $\{-128, \ldots, 127\}$, the silent assumption is that a literal value α larger than 127 converts to $\alpha - 256$, which has the same representation under two's complement.

Chapter 3

Functions

3.1 A first C++ function

Garbage in, garbage out.

Attributed to George Fuechsel, IBM, late 1950's

This section introduces C++ functions as a means to encapsulate and reuse functionality, and to subdivide a program into subtasks. You will learn how to add functions to your programs, and how to call them. We also explain how functions can efficiently be made available for many programs at the same time, through separate compilation and libraries.

In many numerical calculations, computing powers is a fundamental operation (see Section 2.5), and there are many other operations that occur frequently in applications. In C++, *functions* are used to encapsulate such frequently used operations, making it easy to invoke them many times, with different arguments, and from different programs, but *without* having to reprogram them every time.

Even more importantly, functions are used to structure a program. In practice, large programs consist of many small functions, each of which serves a clearly defined subtask. This makes it a lot easier to read, understand, and maintain the program.

We have already seen quite a number of functions, since the main function of every C++ program is a special function (Section 2.1.4).

Program 19 emphasizes the encapsulation aspect and shows how functions can be used. It first defines a function for computing the value b^e for a given real number b and given integer e (possibly negative). It then calls this function for several values of b and e. The computations are performed over the floating point number type double.

```
// Prog: callpow.C
// Define and call a function for computing powers.

#include <iostream>

// PRE:  e >= 0 || b != 0.0
// POST: return value is b^e
double pow (double b, int e)
{
  double result = 1.0;
  if (e < 0) {
    // b^e = (1/b)^(-e)
    b = 1.0/b;
    e = -e;
  }
  for (int i = 0; i < e; ++i) result *= b;
```

```
  return result;
}

int main()
{
  std::cout << pow( 2.0, -2) << "\n"; // outputs 0.25
  std::cout << pow( 1.5,  2) << "\n"; // outputs 2.25
  std::cout << pow( 5.0,  1) << "\n"; // outputs 5
  std::cout << pow( 3.0,  4) << "\n"; // outputs 81
  std::cout << pow(-2.0,  9) << "\n"; // outputs -512

  return 0;
}
```

Program 19: *progs/callpow.C*

Before we explain the concepts necessary to understand this program in detail, let us get an overview of what is going on in the function pow. For nonnegative exponents e, b^e is obtained from the initial value of 1 by e-fold multiplication with b. This is what the for-loop does. The case of negative e can be handled by the formula $b^e = (1/b)^{-e}$: after inverting b and negating e in the if-statement, we have an equivalent problem with a positive exponent. The latter only works if $b \neq 0$, and indeed, negative powers of 0 are mathematically undefined.

3.1.1 Pre- and postconditions

Even a very simple function should document its *precondition* and its *postcondition*, in the form of comments. The precondition specifies what has to hold when the function is called, and the postcondition describes value and effect of the function. This information allows us to understand the function without looking at the actual sourcecode; this in turn is a necessary for keeping track of larger programs. In case of the function pow, the precondition

```
// PRE:  e >= 0 || b != 0.0
```

tells us that e must be nonnegative, or (if e is negative) that $b \neq 0$ must hold. The postcondition

```
// POST: return value is b^e
```

tells us the function value, depending on the arguments. In this case, there is no effect.

The pre- and postconditions specify the function in a mathematical sense. At first sight, functions with values *and* effect [1] do not fit into the framework of mathematical functions which only have values. But using the concept of *program states* (Sec-

[1] Formally, it is the function *call* that has the value and effect, but we suppress this subtlety. Even mathematicians talk about a function value when they mean the value resulting from an evaluation of the function with certain arguments.

tion 1.2.3), a C++ function can be considered as a mathematical function that maps program states (immediately *before* the function call) to program states (immediately *after* the function call).

Under this point of view, the precondition specifies the *domain* of the function, the set of program states in which the function may be called. In case of `pow`, these are all program states in which the arguments `b` and `e` are in a suitable relation. The postcondition describes the function itself by specifying how the (relevant part of the) program state gets transformed. In case of `pow`, the return value b^e will (temporarily) be put at some memory location.

To summarize, the postcondition tells us what happens when the precondition is satisfied. On the other hand, the postcondition gives *no guarantee whatsoever* for the case where the precondition is not satisfied. From a mathematical point of view, this is fine: a function is simply not defined for arguments outside its domain.

Arithmetic pre- and postconditions. The careful reader of Section 2.5 might have realized that both pre- and postcondition of the function `pow` cannot be correct. If e is too large, for example, the computation might overflow, but such e are not excluded by the precondition. Even if there is no overflow, the value range of the type `double` may have a hole at b^e, meaning that this value cannot be returned by the function. The postcondition is therefore imprecise as well.

In the context of arithmetic operations over the fundamental C++ types, it is often tedious and even undesirable to write down precise pre- and postconditions; part of the problem is that fundamental types may behave differently on different platforms. Therefore, we often confine ourselves to pre- and postconditions that document our *mathematical* intention, but we have to keep in mind that in reality, the function might behave differently.

Assertions. So far, our preconditions are just comments like in

```
// PRE:   e >= 0 || b != 0.0
```

Therefore, if the function `pow` is called with arguments `b` and `e` that violate the precondition, this passes unnoticed. On the syntactical level, there is nothing we can do about it: the function call `pow (0.0, -1)`, for example, will compile. But we can make sure that this blunder is detected at runtime. A simple way to do this uses *assertions*. An assertion has the form

assert (*expr*)

where *expr* is a predicate, an expression of a type whose values can be converted to `bool`. No comma is allowed in *expr*, a consequence of the fact that `assert` is not a function but a *macro*. A macro is a piece of meta-code that the compiler replaces with actual C++ code prior to compilation.

With assertions, `pow` can be written as follows.

```
// PRE:  e >= 0 || b != 0.0
// POST: return value is b^e
double pow (double b, int e)
{
  assert (e >= 0 || b != 0.0);
  double result = 1.0;
  // the remainder is as before
  ...
}
```

The purpose of an assertion is to check whether a certain predicate holds at a certain point. The precise semantics of an assertion is as follows. *expr* is evaluated, and if it returns *false*, execution of the program terminates immediately with an error message telling us that the respective assertion was violated. If *expr* returns *true*, execution continues normally. In our case, this means that the evaluation of the expression `pow (0.0,-1)` leads to a *runtime error*. This might not be a very polite way of telling the user that the arguments were illegal but the point will surely come across.

You can argue that it is costly to test the assertion in every function call, just to catch a few "bad" calls. However, it is possible to tell the compiler to ignore the `assert` macro, meaning that an empty piece of C++ code replaces it. The usual way to go is therefore as follows: during code development, put assertions everywhere you want to be sure that something really holds. When the code is stable (and no assertion violations seem to occur anymore), tell the compiler to remove the assertions. The machine language code is then as efficient as if you would never have written the assertions in the first place.

To use the `assert` macro, we have to include the header `cassert`.

3.1.2 Function definitions

Lines 8–18 of Program 19 define a function called `pow`. The syntax of a function definition is as follows.

> *T fname* (*T1 pname1*, *T2 pname2*, ..., *Tk pnamek*)
> *block*

This defines a function called *fname*, with *return type T*, and with *formal arguments pname1*,..., *pnamek* of types *T1*,..., *Tk*, respectively, and with a *function body block*.

Syntactically, *T* and *T1*,..., *Tk* are type names, *fname* as well as *pname1*,..., *pnamek* are identifiers (Section 2.1.9), and *block* is a block, a sequence of statements enclosed by curly braces (Section 2.4.3).

We can think of the formal arguments as placeholders for the actual arguments that are supplied (or "passed") during a function call.

Function definitions must not appear inside blocks, other functions, or control statements. They may appear inside namespaces, though, or at global scope, like in `callpow.C`.

A program may contain an arbitrary number of function definitions, appearing one after another without any delimiters between them. In fact, the program `callpow.C` consists of *two* function definitions, since the `main` function is a function as well.

3.1.3 Function calls

In Program 19, `pow(2.0,-2)` is one of five function calls. Formally, a function call is an expression. The syntax of a function call that matches the general function definition from above is as follows.

fname (*expr1*, ..., *exprk*)

Here, *expr1*,...,*exprk* must be expressions of types whose values can be converted to the formal argument types *T1*,...,*Tk*. These expressions are the *call arguments*. For all types that we know so far, the call arguments as well as the function call itself are rvalues. The type of the function call is the function's return type *T*.

When a function call is evaluated, the call arguments are evaluated first (in an order that is unspecified by the C++ standard). The resulting values are then used to initialize the formal arguments. Finally, the function body is executed; in this execution, the formal arguments behave like they were variables defined in the beginning of *block*, initialized with the values of the call arguments.

The evaluation of a function call terminates as soon as a return statement is reached, see Section 2.1.14. This return statement must be of the form

`return` *expr*;

where *expr* is an expression of a type whose values can be converted to the return type *T*. The resulting value is the value of the function call. The effect of the function call is determined by the joint effects of the call argument evaluations, and of executing *block*.

The function body may contain several return statements, but if no return statement is reached during the execution of *block*, value and effect of the function call are undefined (unless the return type is `void`, see Section 3.1.4 below).

For example, during the execution of *block* in `pow(2.0,-2)`, `b` and `e` initially have values 2 and -2. These values are changed in the `if`-statement to 0.5 and 2, before the subsequent loop sets `result` to 0.5 in its first and to 0.25 in its second and last iteration. This value is returned and becomes the value of the function call expression `pow(2.0,-2)`.

3.1.4 The type void

In C++, there is a fundamental type called `void`, used as return type for functions that only have an effect, but no value. Such functions are also called *void functions*.

As an example, consider the following program (note that the function `print_pair` requires no precondition, since it works for any combination of `int` values).

```
#include <iostream>

// POST: "(i, j)" has been written to standard output
void print_pair (int i, int j)
{
  std::cout << "(" << i << ", " << j << ")\n";
}

int main()
{
  print_pair(3,4); // outputs  (3, 4)
}
```

The type `void` has empty value range, and there are no literals, variables, or formal function arguments of type `void`. There are expressions of type `void`, though, for example `print_pair(3,4)`.

A void function does not require a return statement, but it may contain return statements with *expr* of type `void`, or return statements of the form

```
return;
```

Evaluation of a void function call terminates when a return statement is reached, *or* when the execution of *block* is finished.

3.1.5 Functions and scope

The parenthesized part of a function definition contains the declarations of the formal arguments. For all of them, the declarative region is the function definition, so the formal arguments have local scope (Section 2.4.3). The potential scope of a formal argument declaration begins after the declaration and extends until the end of the function body. Therefore, the formal arguments are not visible outside the function definition. Within the body, the formal arguments behave like variables that are local to *block*.

In particular, changes made to the values of formal arguments (like in the function `pow`) are "lost" after the function call and have no effect on the values of the call arguments. This is not surprising, since the call arguments are rvalues, but to make the point clear, let us consider the following alternative `main` function in `callpow.C`.

```
int main() {
  double b = 2.0;
  int e = -2;
  std::cout << pow(b,e); // outputs 0.25
  std::cout << b;        // outputs 2
  std::cout << e;        // outputs -2
```

```
7
8     return 0;
9   }
```

The values of the variables b and e defined in lines 2–3 stay the same throughout, since the function body of pow is not in the scope of their declarations, for two reasons. First, the definition of pow appears *before* the declarations of b and e in lines 2–3, so the body of pow cannot even be in the *potential* scope of these declarations. Second, even if we would move the declarations of the variables b and e to the beginning of the program (before the definition of pow, so that they have global scope), their scope would exclude the body of pow, since that body is in the potential scopes of redeclarations of the names b and e (the formal arguments), see Section 2.4.3.

But the general scope rules of Section 2.4.3 *do* allow function bodies to use names of global or namespace scope; the program on page 175 for example uses std::cout as such a name. Here is a contrived program that demonstrates how a program may modify a *global variable* (a variable whose declaration has global scope). While such constructions may be useful in certain cases, they usually make the program less readable, since the effect of a function call may then become very non-local.

```
1   #include<iostream>
2
3   int i = 0; // global variable
4
5   void f()
6   {
7     ++i;     // in the scope of declaration in line 3
8   }
9
10  int main()
11  {
12    f();
13    std::cout << i << "\n"; // outputs 1
14
15    return 0;
16  }
```

Since the formal arguments of a function have local scope, they also have automatic storage duration. This means that we get a “fresh” set of formal arguments every time the function is called, with memory assigned to them only until the respective function call terminates.

Names declared inside the function body must be distinct from the names of all formal arguments, unless they appear in a nested block. This makes sense since otherwise, it would be possible to irrevocably hide the name of a formal argument. Therefore, we cannot write

```
int f (int i)
```

```
{
  int i = 5; // invalid; i hides formal argument
  return i;
}
```

while the following is not recommended but legal.

```
int f (int i)
{
  {
    int i = 5; // ok; i is local to nested block
  }
  return i; // the formal argument
}
```

The latter function is the identity, since the scope of the declaration `int i = 5` is limited to the nested block.

Function declarations. A function itself also has a scope, and the function can only be called within its scope. The scope of a function is obtained by combining the scopes of all its *declarations*. The part of the function definition before *block* is a declaration, but there may be function declarations that have no subsequent *block*. This is in contrast to variables where every declaration is at the same time a definition. A function may be declared several times, but it can be defined once only.

The following program, for example, does not compile, since the call of `f` in `main` is not in the scope of `f`.

```
#include<iostream>

int main()
{
  std::cout << f(1); // f undeclared
  return 0;
}

int f (int i) // scope of f begins here
{
  return i;
}
```

But we can put `f` into the scope of `main` by adding a declaration before `main`, and this yields a valid program.

```
#include<iostream>

int f (int i); // scope of f begins here

int main()
```

```
{
  std::cout << f(1); // ok, call is in scope of f
  return 0;
}

int f (int i)
{
  return i;
}
```

In the previous program, we could get rid of the extra declaration by simply defining f before main, but sometimes, separate function declarations are indeed necessary. Consider two functions f and g such that g is called in the function body of f, and f is called in the function body of g. We *have* to define one of the two functions first (f, say), but since we call g within the body of f, g must have a declaration *before* the definition of f.

3.1.6 Procedural programming

So far, we have been able to "live" without functions only since the programs that we have written are pretty simple. But even some of these simple ones would benefit from functions. Consider as an example the program perfect.C from Exercise 44. In this exercise, we have asked you to find the perfect numbers between 1 and n, for a given input number n. The solution so far uses one "big" *double loop* (loop within a loop) that in turn contains two if statements. Although in this case, the "big" loop is still small enough to be read without difficulties, it doesn't really reflect the logical structure of the solution. Once we get to triple or quadruple loops, the program becomes very hard to follow.

But what *is* the logical structure of the solution? For every number i between 1 and n, we have to **test whether i is perfect**; and to do the latter, we have to **compute the sum of all proper divisors of i** and check whether it is equal to i. Thus, we have two clearly defined subtasks that the program has to solve for every number i, and it is best to encapsulate these into functions. Program 20 shows how this is done. Note that the program is now almost self-explanatory: the postconditions can more or less directly be read off the function names.

```
// Program: perfect2.C
// Find all perfect numbers up to an input number n

#include <iostream>

// POST: return value is the sum of all divisors of i
//       that are smaller than i
unsigned int sum_of_proper_divisors (unsigned int i)
```

```
{
  unsigned int sum = 0;
  for (unsigned int d = 1; d < i; ++d)
    if (i % d == 0) sum += d;
  return sum;
}

// POST: return value is true if and only if i is a
//       perfect number
bool is_perfect (unsigned int i)
{
  return sum_of_proper_divisors (i) == i;
}

int main()
{
  // input
  std::cout << "Find perfect numbers up to n =? ";
  unsigned int n;
  std::cin >> n;

  // computation and output
  std::cout << "The following numbers are perfect.\n";
  for (unsigned int i = 1; i <= n ; ++i)
    if (is_perfect (i)) std::cout << i << " ";
  std::cout << "\n";

  return 0;
}
```

Program 20: *progs/perfect2.C*

Admittedly, the program is longer than `perfect.C`, but it is more readable, and it has simpler control flow. In particular, the double loop has disappeared.

The larger a program gets, the more important is it to subdivide it into small subtasks, in order not to lose track of what is going on in the program on the whole; this is the *procedural programming* paradigm, and in C++, it is realized with functions.

The procedural programming paradigm is not so self-evident as it may seem today. The first programming language that became accessible to a general audience since the 1960's was BASIC (**B**eginner's **A**ll-purpose **S**ymbolic **I**nstruction **C**ode).

In BASIC, there were no functions; in order to execute a code fragment responsible for a subtask, you had to use the `GOTO` statement (with a line number)—or `GOSUB` in many dialects—to jump to that code, and then jump back using another `GOTO` (`RETURN`, respectively). The result was often referred to as *spaghetti code*, due to the control flow meandering like a boiled spaghetti on a plate. Moreover, programmers often didn't think in terms of clearly defined subtasks, simply because the language did not support it. This usually lowered the code quality even further.

Despite this, BASIC was an extremely successful programming language. It reached the peak of its popularity in the late 1970's and early 1980's when the proud owners of the first home computers (among them the authors) created programs of fairly high complexity in BASIC. [2]

3.1.7 Arrays as function arguments

We have seen in Section 2.6.2 that an array cannot be initialized from another array, and this implies that arrays have to receive special attention in the context of functions. The usual first step in a function call evaluation (the call arguments are evaluated, and their values are used to initialize the formal arguments) can't be done with arrays.

Given this, it might be surprising that formal arguments of array type are allowed. For example, we could declare a function

```
// PRE:  a[0],...,a[n-1] are elements of an array
// POST: a[i] is set to value, for 0 <= i < n
void fill_n (int a[], int n, int value);
```

to set all elements of an array to some fixed value. The compiler, however, internally *adjusts* this to the completely equivalent declaration

```
// PRE:  a[0],...,a[n-1] are elements of an array
// POST: a[i] is set to value, for 0 <= i < n
void fill_n (int* a, int n, int value);
```

The same adjustment would happen for the formal argument `int a[5]`, say, meaning that the array length is ignored. You could in fact (legally, but quite confusingly) have a formal argument `int a[5]`, and then use an array of length 10 as call argument.

The moral is that in reality, no formal arguments of array type exist, and in order to avoid confusion, it is better not to pretend otherwise.

If we want to build a function that works with arrays, we therefore have to think about alternative ways of passing the array to the function. An obvious way is suggested by the declaration of `fill_n` above: we pass a pointer to the first element, along with the number of elements. A different possibility is to pass *two* pointers, one to the first element, and a past-the-end pointer. This also uniquely describes the array. In both variants, we may actually choose the call arguments in such a way that they describe a contiguous *subarray* of the original array. This generalization is possible since the array itself never appears as an argument.

At this point, it seems like a matter of taste which of the two variants is preferable; but if you think about how the function `fill_n` is naturally implemented in both variants,

[2]If you want to understand how we accomplished this, you should know that we were also passionate enough to type up several pages of program code published in computer magazines, and that we used to store programs as sequences of beeps on audio cassettes.

we see a difference. Here is a program that defines and uses the two variants (the second one is just called `fill` since there is no `n`) that naturally result, given the respective formal arguments.

```
// Program: fill.C
// define and use two functions to fill an array

#include<iostream>

// PRE:  a[0],...,a[n-1] are elements of an array
// POST: a[i] is set to value, for 0 <= i < n
void fill_n (int* a, int n, int value) {
  // iteration by index
  for (int i = 0; i < n; ++i)
    a[i] = value;
}

// PRE:  [first, last) is a valid range
// POST: *p is set to value, for p in [first, last)
void fill (int* first, int* last, int value) {
  // iteration by pointer
  for (int* p = first; p != last; ++p)
    *p = value;
}

int main()
{
  int a[5];
  fill_n (a, 5, 0); // a == {0, 0, 0, 0, 0}
  fill (a, a+5, 1); // a == {1, 1, 1, 1, 1}
  return 0;
}
```

Program 21: *progs/fill.C*

In the first variant, we iterate over all indices in the set $\{0, \ldots, n-1\}$, and we get the array elements by random access. In the second variant, we iterate over all addresses in the *range* `[first, last)`, and we get the array elements by dereferencing. A valid range contains the addresses of a (possibly empty) set of consecutive array elements, where the halfopen interval notation `[first, last)` means that the range is given by the values of `first`, `first+1`, ...,`last-1`. In other words, `last` is a past-the-end pointer for the subarray described by the range.

As we have already argued in Section 2.6.5, the second variant implements the "natural" iteration over an array, and therefore seems preferable. But the *real* reason why it is indeed preferable lies somewhere else. In C++, there are techniques to make functions like fill or fill_n available not only for arrays, but for many other containers at the same time. In this general setting, the functions work with iterators.We may think of them as generalized pointers to container elements, but the operations that we can actually perform on these "pointers" depend on the container.

There are many natural containers that do not offer random access to their elements. For such containers, fill_n as above won't work, since the subscript operator is not available for their "pointers". The underlying operation of adding integers to such "pointers" is then not defined, either.

On the other hand, the way we have defined a container in Section 2.6.5, we *are* guaranteed that we can iterate over its elements. By convention, this is realized through "pointer" increment, using the operator ++. In fact, the operation ++p is available, even if p+1 is not; the latter is random access functionality for the special right-hand side operand 1.

Therefore, the function fill as above has the potential to work for all containers, since it only requires "pointer" functionality that is offered by all container iterators.

Mutating functions. There is a substantial difference between the function pow on the one hand, and the functions fill and fill_n on the other hand. A call to the function pow has no effect, since the computations only modify formal argument values; these values are "local" to the function call and "disappear" upon termination. With pow, it's the *value* of a function call that we are interested in.

Calls to the functions fill and fill_n, on the other hand, have effects: they modify the values of array elements, and these values are *not* local to the function call. When we write

```
int a[5];
fill (a, a+5, 0);
```

the effect of the expression fill (a, a+5, 0) is that all elements of a receive value 0. This is possible since there are formal arguments of pointer type. When the function call fill (a, a+5, 0) is evaluated, the formal argument first is initialized with the *address* of a's first element. In the function body, the value at this address is modified through the lvalue *p, and the same happens for the other four array elements in turn.

Formal arguments of pointer type are therefore a means of constructing functions with value-modifying effects. Such functions are called *mutating*.

3.1.8 Modularization

There are functions that are tailor-made for a specific program, and it would not make sense to use them in another program. But there are also general purpose functions that are useful in many programs. It is clearly undesirable to copy the corresponding function definition into any program that calls the function; what we need is *modularization*, a subdivision of the program into independent parts.

The power function pow from Program 19 is certainly general purpose. In order to make it available to all our programs, we can simply put the function definition into a separate sourcecode file pow.C, say, in our working directory.

```
#include <cassert>

// PRE:  e >= 0 || b != 0.0
// POST: return value is b^e
double pow (double b, int e)
{
  assert (e >= 0 || b != 0.0);
  double result = 1.0;
  if (e < 0) {
    // b^e = (1/b)^(-e)
    b = 1.0/b;
    e = -e;
  }
  for (int i=0; i<e; ++i) result *= b;
  return result;
}
```

Program 22: *progs/pow.C*

Then we can include this file from our main program as follows.

```
// Prog: callpow2.C
// Call a function for computing powers.

#include <iostream>
#include "pow.C"

int main()
{
  std::cout << pow( 2.0, -2) << "\n"; // outputs 0.25
  std::cout << pow( 1.5,  2) << "\n"; // outputs 2.25
  std::cout << pow( 5.0,  1) << "\n"; // outputs 5
  std::cout << pow( 3.0,  4) << "\n"; // outputs 81
  std::cout << pow(-2.0,  9) << "\n"; // outputs -512

  return 0;
}
```

Program 23: *progs/callpow2.C*

An include directive of the form

```
#include "filename"
```

logically replaces the include directive by the contents of the specified file. Usually, *filename* is interpreted relative to the working directory.

Separate compilation and object code files. The code separation mechanism from the previous paragraph has one major drawback: the compiler does not "see" it. Before compilation, pow.C is logically copied back into the main file, so the compiler still has to translate the function definition into machine language *every time* it compiles a program that calls pow. This is a waste of time that can be avoided by *separate compilation*.

In our case, we would compile the file pow.C separately. We only have to tell the compiler that it should not generate an executable program (it can't, since there is no main function) but an *object code* file, called pow.o, say. This file contains the machine language instructions that correspond to the C++ statements in the function body of pow.

Header files. The separate compilation concept is more powerful than we have seen so far: surprisingly, even programs that call the function pow can be compiled separately, without knowing about the source code file pow.C or the object code file pow.o. What the compiler needs to have, though, is a declaration of the function pow.

This function declaration is best put into a separate file as well. In our case, this file pow.h, say, is very short; it contains just the lines

```
// PRE:  e >= 0 || b != 0.0
// POST: return value is b^e
double pow (double b, int e);
```

Since this is the "header" of the function pow, the file pow.h is called a *header file*. In the calling Program 23, we simply replace the inclusion of pow.C by the inclusion of pow.h, resulting in the following program.

```
// Prog: callpow3.C
// Call a function for computing powers.

#include <iostream>
#include "pow.h"

int main()
{
  std::cout << pow( 2.0, -2) << "\n"; // outputs 0.25
  std::cout << pow( 1.5,  2) << "\n"; // outputs 2.25
  std::cout << pow( 5.0,  1) << "\n"; // outputs 5
  std::cout << pow( 3.0,  4) << "\n"; // outputs 81
  std::cout << pow(-2.0,  9) << "\n"; // outputs -512

```

```
  return 0;
}
```

Program 24: *progs/callpow3.C*

From this program, the compiler can then generate an object code file `callpow3.o`. Instead of the machine language instructions for executing the body of `pow`, this object code contains a *placeholder* for the location under which these instructions are to be found in the executable program. It is important to understand that `callpow3.o` cannot be an executable program yet: it *does* contain machine language code for `main`, but *not* for another function that it needs, namely `pow`.

The linker. Only when an executable program is built from `callpow3.o`, the object code file `pow.o` comes into play. Given all object files that are involved, the *linker* builds the executable program by gluing together machine language code for function calls (in `callpow3.o`) with machine language code for the corresponding function bodies (in `pow.o`). Technically, this is done by putting all object files together into a single executable file, and by filling placeholders for function body locations with the actual locations in the executable.

Separate compilation is very useful. It allows to change the definition of a function without having to recompile a single program that calls it. As long as the function declaration remains unchanged, it is only the linker that has to work in the end; and the linker is usually very fast. It follows that separate compilation also makes sense for functions that are specific to one program only.

Separate compilation reflects the "customer" view of the calling program: as long as a function does what its pre- and postcondition promise in the header file, it is not important to know *how* it does this. On the other hand, if the function definition is hidden from the calling program, clean pre- and postconditions are of critical importance, since they may be the only information available about the function's behavior.

Availability of sourcecode. If you have carefully gone through what we have done so far, you realize that we could in principle delete the sourcecode file `pow.C` after having generated `pow.o`, since later, the function definition is not needed anymore. When you buy commercial software, you are often faced with the absence of sourcecode files, since the vendor does not want customers to modify the sourcecode instead of buying updates, or to discover how much money they have paid for lousy software design. [3]

In academic software, availability of sourcecode goes without saying. In order to evaluate or reproduce the contribution of such software to the respective area of research, it is necessary to have sourcecode. Even in commercial contexts, *open source* software is advancing. The most prominent software that comes with all sourcecode files is the operating system *Linux*. Open source software can very efficiently be adapted and

[3]To be fair, we want to remark that there are also more honest reasons for not giving away sourcecode.

improved if many people contribute. But such a contribution is possible only when the sourcecode is available.

Libraries. The function pow will not be the only mathematical function that we want to use in our programs. To make the addition of new functions easy, we can put the definition of pow (and similar functions that we may add later) into a single sourcecode file math.C, say, and the corresponding declarations into a single header file math.h. The object code file math.o then contains machine language code for all our mathematical functions.

Although not strictly necessary, it is good practice to include math.h in the beginning of math.C. This ensures consistency between function declarations and function definitions and puts the code in math.C into the scope of all functions declared in math.h, see Section 3.1.5. In all function bodies in math.C, we can therefore call the other functions, without having to think about whether these functions have already been declared.

In general, several object code files may be needed to generate an executable program, and it would be cumbersome to tell the linker about all of them. Instead, object code files that logically belong together can be *archived* into a *library*. Only the name of this library must then be given to the linker in order to have all library functions available for the executable program. In our case, we so far have only one object file math.o resulting from math.C, but we can still build a library file libmath.a, say, from it.

Figure 17 schematically shows how object code files, a library and finally an executable program are obtained from a number of sourcecode files.

Centralization and namespaces It is clear that we do not want to keep header files and libraries of general interest in our working directory, since we (and others) may have many working directories. Header files and libraries should be at some central place.

We can make our programs independent from the location of header files by writing

```
#include <filename>
```

but in this case, we have to tell the compiler (when we start it) where to search for files to be included. This is exactly the way that headers like iostream from the standard library are included; their locations are known to the compiler, so we don't have to provide any information here. Similarly, we can tell the linker where the libraries we need are to be found. Again, for the various libraries of the standard library, the compiler knows this.

We want to remark that *filename* is not necessarily the name of a physical file; the mapping of *filename* to actual files is implementation defined.

Finally, it is good practice to put all functions of a library into a namespace, in order to avoid clashes with user-declared names, see Section 2.1.3. Let us use the namespace ifm here.

Here are the header and implementation files math.h and math.C that result from these guidelines for our intended library of mathematical functions (that currently contains pow only).

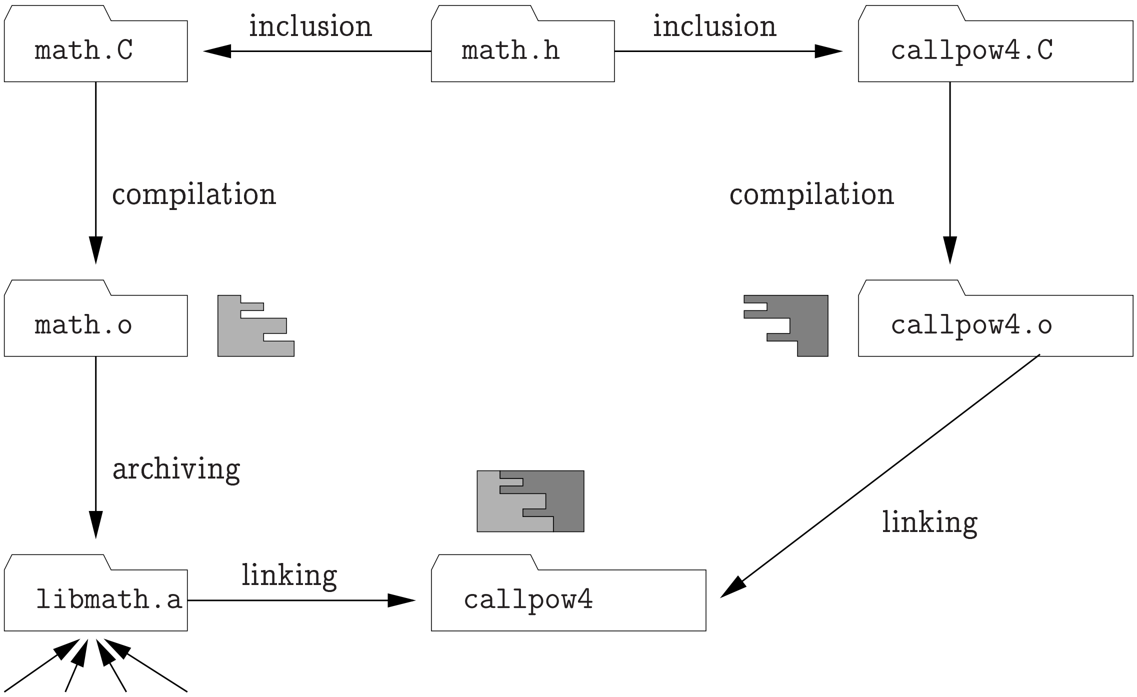


Figure 17: *Building object code files, libraries and executable programs.*

```
// math.h
// A small library of mathematical functions.

namespace ifm {
  // PRE:  e >= 0 || b != 0.0
  // POST: return value is b^e
  double pow (double b, int e);
}
```

Program 25: *progs/math.h*

```
// math.C
// A small library of mathematical functions.

#include <cassert>
#include <IFM/math.h>

namespace ifm {

  double pow (double b, int e)
  {
```

```
    assert (e >= 0 || b != 0.0);
    // PRE:  e >= 0 || b != 0.0
    // POST: return value is b^e
    double result = 1.0;
    if (e < 0) {
      // b^e = (1/b)^(-e)
      b = 1.0/b;
      e = -e;
    }
    for (int i=0; i<e; ++i) result *= b;
    return result;
  }

}
```

Program 26: *progs/math.C*

Finally, the program `callpow4.C` calls our library function `ifm::pow`. It includes the header file `math.h` from a central directory `IFM`.

```
// Prog: callpow4.C
// Call library function for computing powers.

#include <iostream>
#include <IFM/math.h>

int main()
{
  std::cout << ifm::pow( 2.0, -2) << "\n"; // outputs 0.25
  std::cout << ifm::pow( 1.5,  2) << "\n"; // outputs 2.25
  std::cout << ifm::pow( 5.0,  1) << "\n"; // outputs 5
  std::cout << ifm::pow( 3.0,  4) << "\n"; // outputs 81
  std::cout << ifm::pow(-2.0,  9) << "\n"; // outputs -512

  return 0;
}
```

Program 27: *progs/callpow4.C*

3.1.9 Using library functions

You can imagine that we were not the first to put a function like `pow` into a library. Indeed, the standard library contains a function `std::pow` that is even more general than ours: it can compute b^e for *real* exponents e. Accordingly, the arguments of `std::pow` and its return value are of type `double`. In order to use this function, we have to include the header `cmath`. This header contains declarations for a variety of other numerical functions.

Using functions from the standard library can help us to get shorter, better, or more efficient code, without having to write a single new line by ourselves. For example, computing *square roots* can speed up our primality test in Program 8. You might have realized this much earlier, but when we are looking for some proper divisor of a natural number $n \geq 2$, it is sufficient to search in the range $\{2, \ldots, \lfloor\sqrt{n}\rfloor\}$. Indeed, if n can be written as a product $n = dd'$, then the smaller of d and d' must be bounded by $\sqrt{n}$; since the divisors are integral, we even get a bound of $\lfloor\sqrt{n}\rfloor$, $\sqrt{n}$ rounded down.

The primality test could therefore be written more efficiently as in Program 28, using the function `std::sqrt` from the library `cmath`, whose argument and return types are `double`.

```
// Program: prime2.C
// Test if a given natural number is prime.

#include <iostream>
#include <cmath>

int main ()
{
  // Input
  unsigned int n;
  std::cout << "Test if n>1 is prime for n =? ";
  std::cin >> n;

  // Computation: test possible divisors d up to sqrt(n)
  unsigned int bound = (unsigned int)(std::sqrt(n));
  unsigned int d;
  for (d = 2; d <= bound && n % d != 0; ++d);

  // Output
  if (d <= bound)
    // d is a divisor of n in {2,...,[sqrt(n)]}
    std::cout << n << " = " << d << " * " << n / d << ".\n";
  else
    // no proper divisor found
    std::cout << n << " is prime.\n";

  return 0;
}
```

Program 28: *progs/prime2.C*

The program is correct: if `d <= bound` still holds after the loop, we have left the loop because the *other* condition `n % d != 0` has failed. This means that we have found a divisor. If `d > bound` holds after the loop, we have tried all possible divisors smaller or equal to `bound` (whose value is $\lfloor\sqrt{n}\rfloor$, since the explicit conversion rounds down, see Section 2.5.3), so we certainly have not missed any divisor. But we have to be a little careful here: our arguments assume that `std::sqrt` works correctly for squares. For example, `std::sqrt(121)` must return 11 (a little more wouldn't hurt), but *not* 10.99998, say. In that latter case, `(unsigned int)(std::sqrt(121))` would have value 10, and by making this our bound, we miss the divisor 11 of 121, erroneously concluding that 121 is prime.

It is generally not safe to rely on some precise semantics of library functions, even if your platform implements floating point arithmetic according to the IEEE standard 754 (see Section 2.5.6). The square root function is special in the sense that the IEEE standard still guarantees the result of `std::sqrt` to be the floating point number closest to the real square root; consequently, our above implementation of the primality test is safe. But similar guarantees do *not* necessarily hold for other library functions.

Also in our second prime number application, *Eratosthenes's Sieve*, we'd better call a standard library function in order to initialize our list of crossed out numbers, instead of doing it ourselves with a loop. For this, we would replace the two lines

```
  for (unsigned int i = 0; i < n; ++i)
    crossed_out[i] = false;
```

of Program 15 with the single line

```
std::fill (crossed_out, crossed_out + n, false);
```

The pre- and postconditions of this standard library function exactly match the ones of our own `fill` function from Page 181. The benefit here is not the saving of one line of code; this saving does not even exist, since we additionally have to `#include <algorithm>` in the beginning of the program.

The benefit is that we eliminate possible sources of error (even a trivial loop has the potential of being wrong), and that we simplify the control flow (see also Section 2.4.8).

3.1.10 Details

Default arguments. Some functions have the property that there are "natural" values for one or more of their formal arguments. For example, when filling an array of underlying type `int`, the value 0 is such a natural value. In such a case, it is possible to specify this value as a *default argument*; this allows the caller of the function to omit the corresponding call argument and let the compiler insert the default value instead. In case of the function `fill` from Program 21, this would look as follows.

```
// PRE:  [first, last) is a valid range
// POST: *p is set to value, for p in [first, last)
void fill (int* first, int* last, int value = 0) {
  // iteration by pointer
```

```
  for (int* p = first; p != last; ++p)
    *p = value;
}
```

This function can now be called with either two or three arguments, as follows.

```
int a[5];
fill (a, a+5);    // means: fill (a, a+5, 0)
fill (a, a+5, 1);
```

In general, there can be default values for any number of formal arguments, but these arguments must be at consecutive positions $i, i+1, \ldots, k$ among the k arguments, for some i. The function can then be called with any number of call arguments between $i-1$ and k, and the compiler automatically inserts the default values for the missing call arguments.

A function may have a separate declaration that specifies default arguments, like in the following declaration of `fill`.

```
// PRE:  [first, last) is a valid range
// POST: *p is set to value, for p in [first, last)
void fill (int* first, int* last, int value = 0);
```

In this case, the actual definition must not repeat the default arguments (the actual rules are a bit more complicated, but this is the upshot).

Function declarations and definitions. A function may have several declarations, even with the same declarative regions (the latter is not allowed for variables, see Section 2.4.3). The purpose of a function declaration is to put subsequent code into the function's scope, and there may be several places where this is necessary.

On the other hand, every function can have only one definition, and this is the one all its declarations refer to.

Function signatures. In function declarations, the formal argument names *pname1*,..., *pnamek* can be omitted.

This makes sense since these names are only needed in the function definition. The important information, namely domain and range of the function, are already specified by the argument types and the return type. All these types together form the *signature* of the function.

In `math.h`, we could therefore equivalently write the declaration

```
double pow (double, int);
```

The only problem is that we need the formal argument names to specify pre- and post-conditions, without going to lengthy formulations involving "the first argument" and "the second argument". Therefore, we usually write the formal argument names even in function declarations.

Mathematical functions. Many of the mathematical functions that are available on scientific pocket calculators are also available from the math library `cmath`. The following table lists some of them. All are available for the three floating point number types `float`, `double` and `long double`.

name	function
`std::abs`	$\|x\|$
`std::sin`	$\sin(x)$
`std::cos`	$\cos(x)$
`std::tan`	$\tan(x)$
`std::asin`	$\sin^{-1}(x)$
`std::acos`	$\cos^{-1}(x)$
`std::atan`	$\tan^{-1}(x)$
`std::exp`	e^x
`std::log`	$\ln x$
`std::log10`	$\log_{10} x$
`std::sqrt`	$\sqrt{x}$

3.1.11 Goals

Dispositional. At this point, you should ...

1) be able to explain the purpose of functions in C++;
2) understand the syntax and semantics of C++ function definitions and declarations;
3) know what the term "procedural programming" means;
4) understand the function `pow` from Program 19 and the functions `fill_n` and `fill` from Program 21;
5) know that formal arguments of pointer type can be used to write array-processing functions, and mutating functions;
6) know why it makes sense to compile function definitions separately, and to put functions into libraries.

Operational. In particular, you should be able to ...

(G1) give two reasons why it is desirable to subdivide programs into functions;

(G2) find pre- and postconditions for given functions, where the preconditions should be as *weak* as possible, and the postconditions should be as *strong* as possible;

(G3) find syntactical and semantical errors in function definitions, and in programs that contain function definitions;

(G4) evaluate given function call expressions;

(G5) write (mutating) functions for given tasks, and write programs for given tasks that use functions;

(G6) subdivide a given task into small subtasks, and write a program for the given task that uses functions to realize the subtasks;

(G7) build a library on your platform, given that you are told the necessary technical details.

3.1.12 Exercises

Exercise 79 *Find pre- and postconditions for the following functions.* (G2)(G4)

a)
```
int f (double i, double j, double k)
{
  if (i > j)
    if (i > k)
      return i;
    else
      return k;
  else
    if (j > k)
      return j;
    else
      return k;
}
```

b)
```
double g (int i, int j)
{
  double r = 0.0;
  for (int k = i; k <= j; ++k)
    r += 1.0 / k;
  return r;
}
```

Exercise 80 *What are the problems (if any) with the following functions? Fix them and find appropriate pre- and postconditions.* (G2)(G3)

a)
```
bool is_even (int i)
{
  if (i % 2 == 0) return true;
}
```

b)
```
double inverse (double x)
{
  double result;
  if (x != 0.0)
    result = 1.0 / x;
  return result;
}
```

Exercise 81 *What is the output of the following program, depending on the input number* `i`*? Describe the output in mathematical terms, ignoring possible over- and underflows.* (G4)

```
#include<iostream>

int f (int i)
{
  return i * i;
}

int g (int i)
{
  return i * f(i) * f(f(i));
}

void h (int i)
{
  std::cout << g(i) << "\n";
}

int main()
{
  int i;
  std::cin >> i;
  h(i);

  return 0;
}
```

Exercise 82 *Find three problems in the following program.* (G3)(G4)

```
#include<iostream>

double f (double x)
{
  return g(2.0 * x);
}

bool g (double x)
{
  return x % 2.0 == 0;
}

void h ()
{
```

```
  std::cout << result;
}

int main()
{
  double result = f(3.0);
  h();

  return 0;
}
```

Exercise 83 *Simplify the program from Exercise 62 by using the library function* `std::pow`*.* (G5)

Exercise 84 *Assume that on your platform, the library function* `std::sqrt` *is not very reliable. For* x *a value of type* `double` *(*$x \geq 0$*), we let* $s(x)$ *be the value returned by* `std::sqrt(`*expr*`)`*, if expr has value* x*, and we assume that we only know that for some positive value* $\varepsilon \leq 1/2$*, the relative error satisfies*

$$\frac{|s(x) - \sqrt{x}|}{\sqrt{x}} \leq \varepsilon, \quad \forall x.$$

How can you change Program 28 such that it correctly works under this relative error bound? You may assume that the floating point number system used on your platform is binary, and that all values of type `unsigned int` *are exactly representable in this system. (This is a theory exercise.)* (G5)

Exercise 85

a) *Write a function*

```
// POST: return value is true if and only if n is prime
bool is_prime (unsigned int n);
```

and use this function in a program to count the number of twin primes *in the range* $\{2, \ldots, 10000000\}$ *(two up to ten millions). A twin prime is a pair of numbers* $(i, i+2)$ *both of which are prime.*

b) *Is the approach of a) the best (most efficient) one to this problem? If you can think of a better approach, you are free to implement it instead of the one outlined in a).*

(G5)

Exercise 86 *The function* pow *in Program 19 needs* $|e|$ *multiplications to compute* b^e*. Change the function body such that less multiplications are performed. You may use the following fact. If* $e \geq 0$ *and* e *has binary representation*

$$e = \sum_{i=0}^{\infty} b_i 2^i,$$

then

$$b^e = \prod_{i=0}^{\infty} \left(b^{2^i}\right)^{b_i}.$$

(G5)

Exercise 87 *Write a program* swap.C *that defines and calls a function for interchanging the values of two* int *objects. The program should have the following structure.*

```
#include<iostream>

// your function definition goes here

int main() {
  // input
  std::cout << "i =? ";
  int i; std::cin >> i;

  std::cout << "j =? ";
  int j; std::cin >> j;

  // your function call goes here

  // output
  std::cout << "Values after swapping: i = " << i
            << ", j = " << j << ".\n";

  return 0;
}
```

Here is an example run of the completed program:

```
i =? 5
j =? 8
Values after swapping: i = 8, j = 5.
```

(G5)

Exercise 88 *Modify the program* sort_array.C *from Exercise 70 in such way that the resulting program* sort_array2.C *defines and calls a function*

```
// PRE: [first, last) is a valid range
// POST: the elements *p, p in [first, last) are
//       in ascending order
void sort (int* first, int* last);
```

to perform the sorting of the array into ascending order. It may be tempting (but not allowed for obvious reasons) to use `std::sort` *or similar standard library functions in the body of the function* `sort` *that is to be defined. It* is *allowed, though, to compare the efficiency of your* `sort` *function with that of* `std::sort` *(which has the same pre- and postconditions and can be used after* `include<algorithm>`*).*

For this exercise, it is desirable (but not strictly necessary) to use pointer increment (`++p`*) as the only operation on pointers (apart from initialization and assignment, of course). If you succeed in doing so, your sorting function has the potential of working for containers that do not offer random access (see also Section 3.1.7).* (G5)

Exercise 89 *A* perpetual calendar *can be used to determine the weekday (Monday, . . . , Sunday) of any given date. You may for example know that the Berlin wall came down on November 9, 1989, but what was the weekday? (It was a Thursday.) Or what is the weekday of the 1000th anniversary of the Swiss confederation, to be celebrated on August 1, 2291? (Quite adequately, it will be a Saturday.)*

a) *The task is to write a program that outputs the weekday (Monday, . . . , Sunday) of a given input date.*

   Identify a set of subtasks to which you can reduce this task. Such a set is not unique, of course, but all individual subtasks should be small (so small that they could be realized with few lines of code). It is of course possible for a subtask in your set to reduce to other subtasks. (Without giving away anything, one subtask that you certainly need is to determine whether a given year is a leap year).

b) *Write a program* `perpetual_calendar.C` *that reads a date from the input and outputs the corresponding weekday. The range of dates that the program can process should start no later than January 1, 1900 (Monday). The program should check whether the input is a legal date, and if not, reject it. An example run of the program might look like this.*

   ```
   day =? 13
   month =? 11
   year =? 2007
   Tuesday
   ```

   To structure your program, implement the subtasks from a) as functions, and put the program together from these functions.

(G5)(G6)

Exercise 90 *Build a library on your platform from the files* `math.h` *and* `math.C` *in Program 25 and Program 26. Use this library to generate an executable program from Program 27.* (G5)(G7)

Exercise 91

a) *Implement the following function and test it. You may assume that the type* `double` *complies with the IEEE standard 754, see Section 2.5.6. The function is only required to work correctly, if the nearest integer is in the value range of the type* `int`*.* (G5)

```
// POST: return value is the integer nearest to x
int round (double x);
```

b) *The postcondition of the function does not say what happens if there are two nearest integers. Specify the behavior of your implementation in the postcondition of your function.* (G2)

c) Add a declaration of your function to the file `math.h` (Program 25) and a definition to `math.C` (Program 26). Build a library from these two files, and rewrite your test function from a) to call the library version of the function `round`. (G7)

Exercise 92 *This is another (not too difficult) one from* Project Euler *(Problem 56,* `http://projecteuler.net/`*). Find natural numbers* $a, b < 100$ *for which* a^b *has the largest cross sum (sum of decimal digits). Let us say upfront that* 99^{99} *is not the answer.*

Write a program `power_cross_sums.C` *that computes the best* a *and* b *(within reasonable time).*

Can you also find the best a, b *up to* $1,000$*?*

3.1.13 Challenges

Exercise 93 *The simplest computer model that is being studied in theoretical computer science is the* deterministic finite automaton *(DFA). Such an automaton is defined over a finite* alphabet Σ *(for example* $\Sigma = \{0, 1\}$*). Then it has a finite set of* states Q*. The main ingredient is the* transition function

$$\delta : Q \times \Sigma \to Q.$$

We can visualize this function as follows: whenever $\delta(q, \sigma) = q'$*, we draw an arrow from state* q *to state* q'*, labeled with* σ*.*

Finally, there is a starting state $s \in Q$ *and a subset* $F \subseteq Q$ *of* accepting states*. Figure Figure 18 depicts a DFA with state set* $Q = \{0, 1, 2\}$*. The starting state is indicated by an arrow coming in from nowhere, and the accepting states are marked with double circles (in this case, there is only one).*

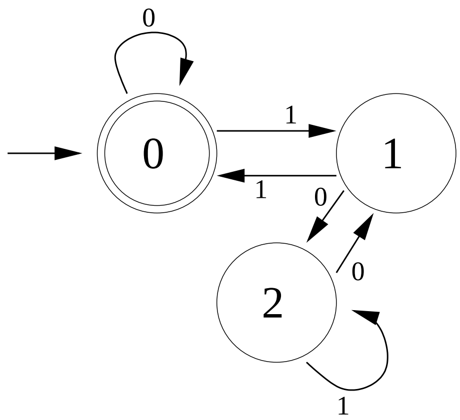


Figure 18: *A deterministic finite automaton (DFA)*

Why can we call such an automaton a computer model? Because it performs a computation, namely the following: given an input word $w \in \Sigma^*$ *(finite sequence of symbols from the alphabet* Σ*), the automaton either* accepts, *or* rejects *it. To do this, the word* w *is processed symbol by symbol, starting in* s*. Whenever the automaton is in some state* q *and the next symbol is* σ*, the automaton switches to state* $q' = \delta(q, \sigma)$*. When all symbols have been processed, the automaton is either in an accepting state* $q \in F$ *(in which case* w *is accepted), or in a non-accepting state* $q \in Q \setminus F$ *(in which case* w *is rejected).*

For example, when we feed the automaton of Figure 18 with the word $w = 0101$*, the sequence of states that are being visited is* $0, 0, 1, 2, 2$*. Consequently,* w *is rejected.*

The language L *of the automaton is the set of accepted words. This is a (generally infinite) subset of* Σ^**. Let's try to determine the language of the automaton in Figure 18.*

It turns out that this is not such a straightforward task, and you need the right idea. (To be honest, we had the idea first and then came up with an automaton that realizes it). We claim that the automaton accepts exactly all the words that are divisible by 3 *if you interpret the word as a binary number (where the empty word is interpreted as* 0*). For example,* 0101 *is the binary number*

$$0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5,$$

and indeed 5 *is not divisible by* 3 *(and hence rejected). In fact (and this is the key to the proof of our claim), the state after processing* w *is the one numbered with* $w \bmod 3$*. You can therefore say that the DFA of Figure 18 is a computer (with a built-in program) that can solve the decision problem of checking whether a given number is divisible by 3.*

We are slowly approaching the actual challenge. For every subset L *of* $\{0, 1\}^*$ *from the following list, either find a DFA that has* L *as its language, or prove* *that such a DFA cannot exist (which would show that DFA are limited in their computational power).*

a) $L = \{w \in \{0,1\}^* \mid w$ *has an even number of zeros and an even number of ones*$\}$

b) $L = \{w \in \{0,1\}^* \mid w$ *is divisible by* 5 *when interpreted as a binary number*$\}$

c) $L = \{w \in \{0,1\}^* \mid w$ *has more zeros than ones*$\}$

d) $L = \{w \in \{0,1\}^* \mid w$ *does not contain three consecutive ones*$\}$

Exercise 94 *A Sudoku puzzle is posed on a grid of* 9×9 *cells, subdivided into 9 square boxes of* 3×3 *cells each. Some grid cells are already filled by numbers between* 1 *and 9; the goal is to fill the remaining cells by numbers between* 1 *and* 9 *in such a way that within each row, column, and box of the completed grid, every number occurs exactly once. Here is an example of a Sudoku puzzle:*

			1			7	4	
	5			9			3	2
		6	7			9		
4			8					
	2						1	
					9			5
		4			7	3		
7	3			2			6	
	6	5			4			

In solving the puzzle, one may try to deduce from the already filled numbers that exactly one *number is a candidate for a suitable empty cell. Then this number is filled into the cell, and the deduction process is repeated. There are two situations where such a deduction for the cell in row* r / *column* c *and number* n *is particularly easy and follows the Sherlock Holmes approach (*How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?*).*

1. *All numbers distinct from* n *already appear somewhere in the same row, column, or 3x3 box. This necessarily means that the cell has to be filled with* n*, since we have eliminated all other* numbers *as impossible.*

2. *All other cells in the same row, or in the same column, or in the same 3x3 box are already known not to contain* n*. Again, the cell has to be filled by* n *then, since we have eliminated all other* cells *for the number* n *within the row, column, or box.*

Write a program `sudoku.C` *that takes as input a Sudoku puzzle in form of a sequence of* 81 *numbers between* 0 *and* 9 *(the grid numbers given row by row, where* 0 *indicates an empty cell). The numbers might be separated by whitespaces, so that the Sudoku puzzle from above could conveniently be encoded like this in a file:*

```
0 0 0  1 0 0  7 4 0
0 5 0  0 9 0  0 3 2
0 0 6  7 0 0  9 0 0

4 0 0  8 0 0  0 0 0
0 2 0  0 0 0  0 1 0
0 0 0  0 0 9  0 0 5

0 0 4  0 0 7  3 0 0
7 3 0  0 2 0  0 6 0
0 6 5  0 0 4  0 0 0
```

The program should now try to solve the puzzle by using only the two Sherlock-Holmes-type deductions from above. The output should be a (partially) completed grid that is either the solution to the puzzle, or the unique (why?) partial solution in which no Sherlock-Holmes-type deductions apply anymore (again, empty cells should be indicated by the digit 0*).*

In the above example, the output of a correct program will be the solution:

```
3 9 2  1 8 5  7 4 6
8 5 7  4 9 6  1 3 2
1 4 6  7 3 2  9 5 8

4 7 9  8 5 1  6 2 3
5 2 8  6 7 3  4 1 9
6 1 3  2 4 9  8 7 5

2 8 4  5 6 7  3 9 1
7 3 1  9 2 8  5 6 4
9 6 5  3 1 4  2 8 7
```

For reading the input from a file, it can be convenient to redirect the standard input to the file containing the puzzle data. For checking whether any Sherlock-Holmes-type deductions apply, it can be useful to maintain (and update) for any triple (r, c, n) *the information whether* n *is still a possible candidate for the cell in row r / column c.*

You will discover that many Sudoku puzzles that typically appear in newspapers can be solved by your program and are therefore easy, even if they are labeled as medium *or* hard.

Hint: *It is advisable not to optimize for efficiency here, since this will only lead to more complicated and error-prone code. Given the very small problem size, such optimizations won't have a noticeable effect anyway.*

3.2 Recursion

Mir san mir.

Bavarian dictum, meaning "we are we".

This section introduces recursive functions, functions that directly or indirectly call themselves. You will see that recursive functions are very natural in many situations, and that they lead to compact and readable code close to mathematical function definitions. We will also explain how recursive function calls are processed, and how recursion can (in principle) be replaced with iteration. In the end, you will see two applications (sorting, and drawing fractals) that demonstrate the power or recursion.

3.2.1 A warm-up

Many mathematical functions are naturally defined *recursively*, meaning that the function to be defined appears in its own definition. For example, for any $n \in \mathbb{N}$, the number $n!$ can recursively be defined as follows.

$$n! := \begin{cases} 1, & \text{if } n \leq 1 \\ n \cdot (n-1)!, & \text{if } n > 1. \end{cases}$$

In C++ we can also have recursive functions: a function may call itself. This is nothing exotic, since after all, a function call is just an expression that can in principle appear anywhere in the function's scope, and that scope includes the function body. Here is a recursive function for computing $n!$; in fact, this definition exactly matches the mathematical definition from above.

```
// POST: return value is n!
unsigned int fac (unsigned int n)
{
  if (n <= 1) return 1;
  return n * fac(n-1); // n > 1
}
```

Here, the expression `fac(n-1)` is a *recursive call* of `fac`.

Infinite recursion. With recursive functions, we have the same issue as with loops (Section 2.4.2): it is easy to write down function calls whose evaluation does not terminate. Here is the shortest way of creating an infinite recursion: define the function

```
void f()
{
  f();
}
```

with no arguments and evaluate the expression `f()`. The reason for non-termination is clear: the evaluation of `f()` consists of an evaluation of `f()` which consists of an evaluation of `f()` which... you get the picture.

Like for loops, the function definition has to make sure that progress towards termination is made in every function call. For the function `fac` above, this is the case: each time `fac` is called recursively, the value of the call argument becomes smaller, and when the value reaches 1, no more recursive calls are performed: we say that the recursion "bottoms out".

3.2.2 The call stack

Let's try to understand what exactly happens during the evaluation of `fac(3)`, say. The formal argument `n` is initialized with 3, and since this is greater than 1, the statement `return n * fac(n-1);` is executed next. This first evaluates the expression `n * fac(n-1)` and in particular the right operand `fac(n-1)`. Since `n-1` has value 2, the formal argument `n` is therefore initialized with 2.

But wait: what is "the" formal argument? Automatic storage duration implies that each function call has its "own" fresh instance of the formal argument, and the lifetime of this instance is the respective function call. In evaluating `f(n-1)`, we therefore get a new instance of the formal argument `n`, on top of the previous instance from the call `f(3)` (that has not yet terminated). But which instance of `n` do we use in the evaluation of `f(n-1)`? Quite naturally, it will be the new one, the one that "belongs" to the call `f(n-1)`. This rule is in line with the general scope rules from Section 2.4.3: the relevant declaration is always the most recent one that is still visible.

The technical realization of this is very simple. Everytime a function is called, the call argument is evaluated, and the resulting value is put on the *call stack* which is simply a region in the computer's memory.[4]

Like a stack of papers on your desk, the call stack has the property that the object that came last is "on top". Upon termination of a function call, the top object is taken off the stack again. Whenever a function call accesses or changes its formal argument, it does so by accessing or changing the corresponding object on top of the stack.

This has all the properties we want: every function call works with its own instance of the formal argument; when it calls another function (or the function itself recursively), this instance becomes temporarily hidden, until the nested call has terminated. At that point, the instance reappears on top of the stack and allows the original function call to work with it again.

[4] if the function has several arguments, several values are put on the call stack; to keep the description simple, we concentrate on the case of one argument.

Table 5 shows how this looks like for `f(3)`, assuming that the right operand of the multiplication operator is always evaluated first. Putting an object on the stack "pushes" it, and taking the top object of "pops" it.

call stack (bottom $\longleftrightarrow$ top)				evaluation sequence	action
···				`fac(3)`	push 3
···	n: 3			`n * fac(n-1)`	
···	n: 3			`n * fac(2)`	push 2
···	n: 3	n: 2		`n * (n * fac(n-1))`	
···	n: 3	n: 2		`n * (n * fac(1))`	push 1
···	n: 3	n: 2	n: 1	`n * (n * 1)`	pop
···	n: 3	n: 2		`n * (2 * 1)`	
···	n: 3	n: 2		`n * 2`	pop
···	n: 3			`3 * 2`	
···	n: 3			`6`	pop
···					

Table 5: *The call stack, and how it evolves during an evaluation of* `fac(3)`*; the respective value of* `n` *to use is always the one on top*

Because of the call stack, infinite recursions do not only consume time but also memory. Unlike infinite loops, they usually lead to a program abortion as soon as the memory reserved for the call stack is full.

3.2.3 Basic practice

Let us consider two more simple recursive functions that are somewhat more interesting than `fac`. They show that recursive functions are particularly amenable to correctness proofs of their postconditions, and this makes them attractive. On the other hand, we also see that it is easy to write innocent-looking recursive functions that are very inefficient to evaluate.

Greatest common divisor. Consider the problem of finding the greatest common divisor $\gcd(a, b)$ of two natural numbers a, b. This is defined as the largest natural number that divides both a and b without remainder. In particular, $\gcd(n, 0) = \gcd(0, n) = n$ for $n > 0$; let us also define $\gcd(0, 0) := 0$.

The *Euclidean algorithm* finds $\gcd(a, b)$, based on the following

Lemma 1 *If* $b > 0$*, then*

$$\gcd(a, b) = \gcd(b, a \bmod b).$$

Proof. Let k be a divisor of b. From

$$a = (a \operatorname{div} b)b + a \bmod b$$

it follows that

$$\frac{a}{k} = (a \operatorname{div} b)\frac{b}{k} + \frac{a \bmod b}{k}.$$

Since $a \operatorname{div} b$ and b/k are integers, we get

$$\frac{a \bmod b}{k} \in \mathbb{N} \quad \Leftrightarrow \quad \frac{a}{k} \in \mathbb{N}.$$

In words, if k is a divisor of b, then k divides a if and only if k divides $a \bmod b$. This means, the divisors of a *and* b are exactly the divisors of b *and* $a \bmod b$. This proves that $\gcd(a, b)$ and $\gcd(b, a \bmod b)$ are equal. □

Here is the corresponding C++ function for computing the greatest common divisor of two `unsigned int` values, according to the Euclidean algorithm.

```
// POST: return value is the greatest common divisor of a and b
unsigned int gcd (unsigned int a, unsigned int b)
{
  if (b == 0) return a;
  return gcd(b, a % b);  // b != 0
}
```

The Euclidean algorithm is very fast. We can easily call it for any `unsigned int` values on our platform, without noticing any delay in the evaluation.

Correctness and termination. For recursive functions, it is often very easy to prove that the postcondition is correct, by using the underlying mathematical definition directly (like $n!$ for `fac`), or by using some facts that follow from the mathematical definition (like Lemma 1 for `gcd`).

The correctness proof must involve a termination proof, so let's start with this: any call to `gcd` terminates, since the value b of the second argument is bounded from below by 0 and gets smaller in every recursive call (we have $a \bmod b < b$).

Given this, the correctness of the postcondition follows from Lemma 1 by induction on b. For $b = 0$, this is clear. For $b > 0$, we inductively assume that the postcondition is correct for all calls to `gcd` where the second argument has value $b' < b$. Since $b' = a \bmod b$ satisfies $b' < b$, we may assume that the call `gcd(b, a % b)` correctly returns $\gcd(b, a \bmod b)$. But by the lemma, $\gcd(b, a \bmod b) = \gcd(a, b)$, so the statement

```
return gcd(b, a % b);
```

correctly returns $\gcd(a, b)$.

Fibonacci numbers. The sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ of *Fibonacci numbers* is one of the most famous sequences in mathematics. Formally, the sequence is defined as follows.

$$\begin{aligned} F_0 &:= 0, \\ F_1 &:= 1, \\ F_n &:= F_{n-1} + F_{n-2}, \quad n > 1. \end{aligned}$$

This means, every element of the sequence is the sum of the two previous ones. From this definition, we can immediately write down a recursive C++ function for computing Fibonacci numbers, getting termination and correctness for free.

```
// POST: return value is the n-th Fibonacci number F_n
unsigned int fib (unsigned int n)
{
  if (n == 0) return 0;
  if (n == 1) return 1;
  return fib(n-1) + fib(n-2); // n > 1
}
```

If you write a program to compute the Fibonacci number F_n using this function, you will notice that somewhere between $n = 30$ and $n = 50$, the program becomes very slow. You even notice how much slower it becomes when you increase n by just 1.

The reason is that the mathematical definition of F_n does not lead to an efficient algorithm, since all values $F_i, i < n-1$, are repeatedly computed, some of them extremely often. You can for example check that the call to `fib(50)` computes F_{48} already twice (once directly in `fib(n-2)`, and once indirectly from `fib(n-1)`. F_{47} is computed three times, F_{46} five times, and F_{45} eight times (do you see a pattern?).

3.2.4 Recursion versus iteration

From a strictly functional point of view, recursion is superfluous, since it can be simulated through iteration (and a call stack explicitly maintained by the program; we could simulate the call stack with an array). We don't have the means to prove this here, but we want to show it for the recursive functions that we have seen in the previous section.

The function `gcd` is very easy to write iteratively, since it is *tail-end recursive*. This means that there is only one recursive call, and that one appears at the very end of the function body. Tail-end recursion can be replaced by a simple loop that iteratively updates the formal arguments until the termination condition is satisfied. In the case of `gcd`, this update corresponds to the transformation $(a, b) \to (b, a \bmod b)$.

```
// POST: return value is the greatest common divisor of a and b
unsigned int gcd2 (unsigned int a, unsigned int b)
{
  while (b != 0) {
    unsigned int a_prev = a;
```

```
    a = b;
    b = a_prev % b;
  }
  return a;
}
```

You see that we get longer and less readable code, and that we need an extra variable to remember the previous value of a before the update step; in the spirit of Section 2.4.8, we should therefore use the original recursive formulation.

Our function fib for computing Fibonacci numbers is not tail-end recursive, but it is still easy to write it iteratively. Remember that F_n is the sum of F_{n-1} and F_{n-2}. We can therefore write a loop whose iteration i computes F_i from the previously computed values F_{i-2} and F_{i-1} that we maintain in the variables a and b.

```
// POST: return value is the n-th Fibonacci number F_n
unsigned int fib2 (unsigned int n)
{
  if (n == 0) return 0;
  if (n <= 2) return 1;
  unsigned int a = 1;  // F_1
  unsigned int b = 1;  // F_2
  for (unsigned int i = 3; i <= n; ++i) {
    unsigned int a_prev = a;  // F_{i-2}
    a = b;                    // F_{i-1}
    b += a_prev;              // F_{i-1} += F_{i-2} -> F_i
  }
  return b;
}
```

Again, this non-recursive version fib2 is substantially longer and more difficult to understand than fib, but this time there is a benefit: fib2 is much faster, since it computes every number $F_i, i \leq n$ *exactly once*. While we would grow old in waiting for the call fib(50) to terminate, fib2(50) gives us the answer in no time. (Unfortunately, this answer may be incorrect, since F_{50} could exceed the value range of the type unsigned int.)

In this case we would prefer fib2 over fib, simply since fib is too inefficient for practical use. The more complicated function definition of fib2 is a moderate price to pay for the speedup that we get.

3.2.5 Primitive recursion

Roughly speaking, a mathematical function is *primitive recursive* if it can be written as a C++ function f in such a way that f neither directly nor indirectly calls itself with call arguments depending on f. For example,

```
unsigned int f (unsigned int n)
```

```
{
  if (n == 0) return 1;
  return f(f(n-1) - 1);
}
```

is not allowed, since f recursively calls itself with a argument depending of f. This does *not* mean that the underlying mathematical function is not primitive recursive, it just means that we have taken the wrong C++ function. Indeed, the above f implements the mathematical function satisfying $f(n) = 1$ for all n, and this function is obviously primitive recursive.

In the early 20-th century, it was believed that the functions whose values can in principle be computed by a machine are exactly the primitive recursive ones. Indeed, the function values one computes in practice (including $\gcd(a, b)$ and F_n) come from primitive recursive functions.

It later turned out that there are computable functions that are not primitive recursive. A simple and well-known example is the binary *Ackermann function* $A(m, n)$, defined by

$$A(m,n) = \begin{cases} n+1, & \text{if } m = 0 \\ A(m-1,1), & \text{if } m > 0, n = 0 \\ A(m-1, A(m, n-1)), & \text{if } m > 0, n > 0. \end{cases}$$

The fact that this function is not primitive recursive requires a proof (that we don't give here). As already noted above, it is necessary but not sufficient that this definition recursively uses A with a argument that depends on A.

It may not be immediately clear that the corresponding C++ function

```
// POST: return value is the Ackermann function value A(m,n)
unsigned int A (unsigned int m, unsigned int n) {
  if (m == 0) return n+1;
  if (n == 0) return A(m-1,1);
  return A(m-1, A(m, n-1));
}
```

always terminates, but Exercise 96 asks you to show this. Table 6 lists some Ackermann function values. For $m \leq 3$, $A(m, n)$ looks quite moderate, but starting from $m = 4$, the values get extremely large. You can still compute $A(4, 1)$, although this takes surprisingly long already. You *might* be able to compute $A(4, 2)$; after all, $2^{65536}-3$ has "only" around $20,000$ decimal digits. But the call to A(4,3) will not terminate within any observable period.

It can in fact be shown that $A(n, n)$ grows faster than any primitive recursive function in n (and this is a proof that A cannot be primitive recursive). Recursion is a powerful but also dangerous tool, since it is easy to encode (too) complicated computations with very few lines of code.

		n					
		0	1	2	3	$\cdots$	n
	0	1	2	3	4	$\cdots$	$n+1$
	1	2	3	4	5	$\cdots$	$n+2$
m	2	3	5	7	9	$\cdots$	$2n+3$
	3	5	13	29	61	$\cdots$	$2^{n+3}-3$
	4	13	65533	$2^{65536}-3$	$2^{2^{65536}}-3$	$\cdots$	$\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n+3}-3$

Table 6: *Some values of Ackermann's function*

3.2.6 Sorting

Sorting a sequence of values (numbers, texts, etc.) into ascending order is a very basic and important operation. For example, a specific value can be found much faster in a sorted than in an unsorted sequence (see Exercise 102). You know this from daily life, and that's why you sort your CDs, and why the entries in a telephone directory are sorted by name.

We have asked you in Exercise 70 to write a program that sorts a given sequence of integers; Exercise 88 was about making this into a function that sorts all numbers described by a given pointer range. In both exercises, you were not supposed to do any efficiency considerations.

Here we want to catch up on this and investigate the *complexity* of the sorting problem. Roughly speaking, the complexity of a problem is defined as the complexity (runtime) of the fastest algorithm that solves the problem. In computing Fibonacci numbers in Section 3.2.3 and Section 3.2.4, we have already seen that the runtimes of different algorithms for the same problem may vary a lot. The same is true for sorting algorithms, as we will discover shortly.

Let us start by analyzing one of the "obvious" sorting algorithms that you may have come up with in Exercise 70. The simplest one that the authors can think of is *minimum-sort*. Given the sequence of values (let's assume they are integers), *minimum-sort* first finds the smallest element of the sequence; then it interchanges this element with the first element. The sequence now starts with the smallest element, as desired, but the remainder of the sequence still needs to be sorted. But this is done in the same way: the smallest element among the remaining ones is found and interchanged with the *second* element of the sequence, and so on.

Assuming that the sequence is described by a pointer range `[first, last)`, *minimum-sort* can be realized as follows.

```
// PRE: [first, last) is a valid range
// POST: the elements *p, p in [first, last) are in ascending order
void minimum_sort (int* first, int* last)
{
```

```
  for (int* p = first; p != last; ++p) {
    // find minimum in nonempty range described by [p, last)
    int* p_min = p; // pointer to current minimum
    int* q = p;     // pointer to current element
    while (++q != last)
      if (*q < *p_min) p_min = q;
    // interchange *p with *p_min
    std::iter_swap (p, p_min);
  }
}
```

The standard library function `std::iter_swap` interchanges the values of the objects pointed to by its two arguments. There is also a function `std::min_element` that we could use to get rid of the inner loop; however, since we want to analyze the function `minimum_sort` in detail, we refrain from calling any nontrivial standard library function here.

What can we say about the runtime of `minimum_sort` for a given range? That it depends on the platform, this is for sure. On a modern PC, the algorithm will run much faster than on a vintage computer from the twentieth century. There is no such thing as "the" runtime. But if we look at what the algorithm does, we can find a measure of runtime that is platform-independent.

A dominating operation in the sense that it occurs very frequently during a call to `minimum_sort` is the comparison `*q < *p_min`. We can even exactly count the number of such comparisons, depending on the number of elements n that are to be sorted. In the first execution of the `while` statement, the first element is compared with all $n-1$ succeeding elements. In the second execution, the second element in compared with all the $n-2$ succeeding elements, and so on. In the second-to-last execution of the `while` statement, finally, we have one comparison, and that's it. We therefore have the following

Observation 1 *The function* `minimum_sort` *sorts a sequence of* n *elements with*

$$1+2+\ldots n-1=\frac{n(n-1)}{2}$$

comparisons between sequence elements.

Why do we specifically count these comparisons? Because any other operation is either performed much less frequently (for example, the declaration statement `int* q = p` is executed only n times), or with approximately the same frequency. This concerns the assignment `p_min = q` which may happen up to $n(n-1)/2$ times, and the expression `++q != last`; this one is evaluated even more frequently, namely $n(n-1)/2+n$ times. The total number of operations is therefore at most $c_1 n(n-1)/2+c_2 n$ for some constants c_1, c_2. For large n, the linear term $c_2 n$ is negligible compared to the quadratic term $c_1 n(n-1)/2$; we can therefore conclude that the total number of operations needed to sort n numbers is proportional to the number of comparisons between sequence elements.

This implies the following: if you measure the runtime of the whole sorting algorithm, the resulting time T_{total} will be proportional to the time T_{comp} that is being spent with comparisons between sequence elements.[5] Since T_{comp} is in turn proportional to the number of comparisons itself, this number is a good indicator for the efficiency of the algorithm.

If you think about sorting more complicated values (like names in a telephone directory), a comparison between two elements might even become the single most time-consuming operation. In such a scenario, T_{comp} may eat up almost everything of T_{total}, making the comparison count an even more appropriate measure of efficiency.

To check that all this is not only grey theory, let us make some experiments and measure the time that it takes to execute a program with the following `main` function, for various values of n. As our "test case", we use the jumbled sequence $0, n-1, 1, n-2, ...$, and after having called the function `minimum_sort` from above, we check whether we now indeed have the ascending sequence $0, 1, \ldots, n-1$. Yes, this program does other things apart from the actual sorting, but all additional operations are "cheap" in the sense that their number is proportional to n at most; according to our above line of arguments, they should therefore not matter.

```
int main()
{
  int n = 100000; // number of values to be sorted
  int* a = new int[n];

  std::cout << "Sorting " << n << " integers...\n";

  // create sequence: 0, n-1, 1, n-2,...
  for (int i=0; i<n; ++i)
    if (i % 2 == 0) a[i] = i/2; else a[i] = n-1-i/2;

  // sort into ascending order
  minimum_sort (a, a+n);

  // is it really sorted ?
  for (int i=0; i<n-1;++i)
    if (a[i] != i) std::cout << "Sorting error!\n";

  delete[] a;

  return 0;
}
```

Table 7 summarizes the results. For every value of n, **Gcomp** is the number of Gigacomparisons (10^9 comparisons), according to Observation 1. In other words, **Gcomp**=

[5]Due to the effects of caching and other add-ons to the von-Neumann architecture, this is not necessarily true on your platform.

n	100,000	200,000	400,000	800,000	1,600,000
Gcomp	5	20	80	320	1280
Time (min)	0:17	1:07	4:24	18:06	73:32
sec/Gcomp	3.4	3.35	3.3	3.4	3.45

Table 7: *Number of comparisons and runtime of* minimum-sort

$10^{-9}n(n-1)/2$. **Time** is the absolute runtime of the program in minutes and seconds, on a modern PC. **sec/Gcomp** is **Time** (in seconds) divided by **Gcomp** and tells us how many seconds the program needs to perform one Gigacomparison.

The table shows that the number of seconds per Gigacomparison is around 3.4 for all considered values of n. As predicted above, the runtime in practice is therefore indeed proportional to the number of comparisons between sequence elements. This number quadruples from one column to the next, and so does the runtime.

We also see that sorting numbers using *minimum-sort* appears to be pretty inefficient. 1,600,000 is not large by today's standards, but to sort that many numbers takes more than one hour! Given that **sec/Gcomp** appears to be constant, we can even estimate the time that it would take to sort 10,000,000 numbers, say. For this, we derive from Observation 1 the required number of Gigacomparisons (50,000) and multiply it with 3.4. The resulting 170,000 seconds are almost two days.

Essentially the same figures result from running other well-known simple sorting algorithms like *bubble-sort* or *insert-sort*. Can we do better? Yes, we can, and recursion helps us to do it!

Merge-sort. The paradigm behind the *merge-sort* algorithm is this: if a problem is (too) large to be solved directly, subdivide it into smaller subproblems that are easier to solve, and then put the overall solution together from the solutions of the subproblems. This paradigm is known as *divide and conquer*.

Here is how this works for sorting. Let us imagine that the numbers to be sorted come as a deck of cards, with the numbers written on them. The first step is to partition the deck into two smaller decks of half the size each. These two decks are then sorted independently from each other, with the same method; but the main ingredient of this method comes only now: we have to merge the two sorted decks into one sorted deck. But this is not hard: we put the two decks in front of us (both now have their smallest card on top); as long as there are still cards in one or both of the decks, the smaller of the two top cards (or the single remaining top card) is taken off and put upside down on a new deck that in the end represents the result of the overall sorting process. Figure 19 visualizes the merge step.

Here is how *merge-sort* can be realized in C++, assuming that we have a function `merge` that performs the above operation of merging two sorted sequences into one sorted sequence.

```
// PRE: [first, last) is a valid range
```

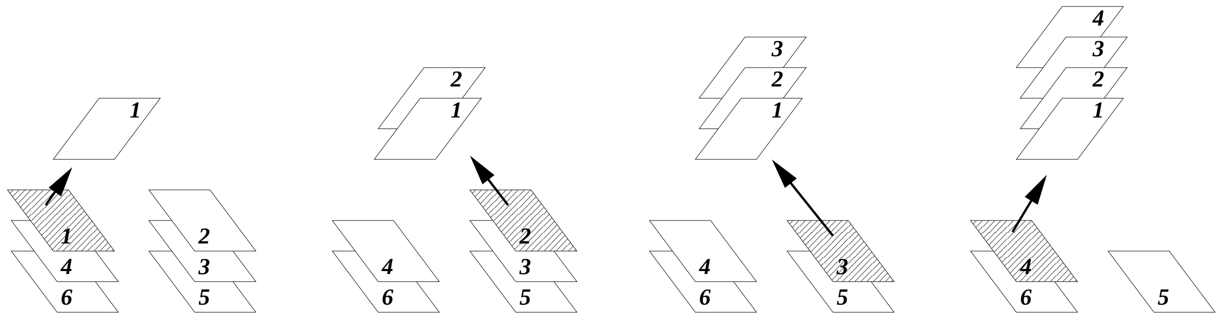


Figure 19: *Merging two sorted decks of cards into one sorted deck*

```
// POST: the elements *p, p in [first, last) are in ascending order
void merge_sort (int* first, int* last)
{
  int n = last - first;
  if (n <= 1) return;          // nothing to do
  int* middle = first + n/2;
  merge_sort (first, middle);  // sort first half
  merge_sort (middle, last);   // sort second half
  merge (first, middle, last); // merge both halfs
}
```

If there is more than one element to sort, the function splits the range `[first, last)` into two ranges `[first, middle)` and `[middle, last)` of lengths $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Just as a reminder, for any real number x, $\lceil x \rceil$ is the smallest integer greater or equal to x ("x rounded up"), and $\lfloor x \rfloor$ is the largest integer smaller or equal to x ("x rounded down"). If n is even, both values $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ are equal to $n/2$, and otherwise, the first value is smaller by one.

As its next step, the algorithm recursively sorts the elements described by both ranges. In the end, it calls the function `merge` on the two ranges. In commenting the latter function, we stick to the deck analogy that we have used above. If you have understood the deck merging process, you will perceive the definition of `merge` as being straightforward.

```
// PRE: [first, middle), [middle, last) are valid ranges; in
//      both of them, the elements are in ascending order
void merge (int* first, int* middle, int* last)
{
  int n = last - first;    // total number of cards
  int* deck = new int[n]; // new deck to be built

  int* left = first;     // top card of left deck
  int* right = middle; // top card of right deck
  for (int* d = deck; d != deck + n; ++d)
    // put next card onto new deck
```

```
    if        (left == middle) *d = *right++; // left deck is empty
    else if (right == last)  *d = *left++;  // right deck is empty
    else if (*left < *right) *d = *left++;  // smaller top card left
    else                     *d = *right++; // smaller top card right

  // copy new deck back into [first, last)
  int *d = deck;
  while (first != middle) *first++ = *d++;
  while (middle != last) *middle++ = *d++;

  delete[] deck;
}
```

Analyzing merge-sort. As for *minimum-sort*, we will count the number of comparisons between sequence elements that occur when a sequence of n numbers is being sorted. Again, we can argue that the total number of operations is proportional to this number of comparisons. For *merge-sort*, this fact is not so immediate, though, and we don't expect you to understand it now. But for the benefit of (not only) the sceptic reader, we will check this fact experimentally below, as we did for *minimum-sort*.

All the comparisons take place during the calls to the function `merge` at the various levels of recursion, so let us first count the number of comparisons between sequence elements that one call to `merge` performs in order to create a sorted deck of n cards from two sorted decks.

It is apparent from the function body (and also from our informal description of the merging process above) that *at most one* comparison is needed for every card that is put on the new deck. Indeed, we may have to compare the two top cards of the left and the right deck in order to find out which card to take off next. But if one of the two decks becomes empty (this situation definitely occurs before the last card is put on the new deck), we don't do any further comparisons. This means that *at most* $n-1$ comparisons between sequence elements are performed in merging two sorted decks into one sorted deck with n cards.

Knowing this, we can now prove our main result.

Theorem 2 *The function* `merge_sort` *sorts a sequence of* $n \geq 1$ *elements with at most*

$$(n-1)\lceil \log_2 n \rceil$$

comparisons between sequence elements.

Proof. We define $T(n)$ to be the *maximum* possible number of comparisons between sequence elements that can occur during a call to `merge_sort` with an argument range of length n. For example, $T(0) = T(1) = 0$, since for ranges of lengths 0 and 1, no comparisons are made. We also get $T(2) = 1$, since for a range of length 2, *merge-sort* performs one comparison (in merging two sorted decks of one card each into one sorted deck of two cards). In a similar way, we can convince ourselves that $T(3) = 2$. There *are* sequences of length 3 for which one comparison suffices (the first card may be taken off the left deck which consists only of one card), but the maximum number that defines $T(3)$ is 2.

For general $n \geq 2$, we have the following recurrence relation:

$$T(n) \leq T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + n - 1. \tag{3.1}$$

To see this, let us consider a sequence of n elements that actually requires the maximum number of $T(n)$ comparisons. This number of comparisons is the sum of the respective numbers in sorting the left and the right half, plus the number of comparisons during the merge step. The former two numbers are (by construction of `merge_sort` and definition of T) at most $T(\lfloor n/2 \rfloor)$ and $T(\lceil n/2 \rceil)$, while the latter number is at most $n-1$ by our previous considerations regarding `merge`. It follows that $T(n)$, the actual number of comparisons, is bounded by the sum of all three numbers.

Now we can prove the actual statement of the theorem. Since the *merge-sort* algorithm is recursive, it is natural that the proof is inductive. For $n = 1$, we have $T(1) = 0 = (1-1)\lceil \log_2 2 \rceil$, so the statement holds for $n = 1$.

For $n \geq 2$, let us assume that the statement of the theorem holds for *all* values in $\{1, \ldots, n-1\}$ (this is the inductive hypothesis). From this hypothesis, we need to derive the validity of the statement for the number n itself (note that $\lfloor n/2 \rfloor, \lceil n/2 \rceil \geq 1$). This goes as follows.

$$\begin{array}{rcll}
T(n) & \leq & T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + n - 1 & \text{(Equation 3.1))} \\
& \leq & (\lfloor \frac{n}{2} \rfloor - 1)\lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil + (\lceil \frac{n}{2} \rceil - 1)\lceil \log_2 \lceil \frac{n}{2} \rceil \rceil + n - 1 & \text{(inductive hypothesis)} \\
& \leq & (\lfloor \frac{n}{2} \rfloor - 1)(\lceil \log_2 n \rceil - 1) + (\lceil \frac{n}{2} \rceil - 1)(\lceil \log_2 n \rceil - 1) + n - 1 & \text{(Exercise 103)} \\
& = & (n-2)(\lceil \log_2 n \rceil - 1) + n - 1 & (\ n = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil) \\
& \leq & (n-1)(\lceil \log_2 n \rceil - 1) + n - 1 & \\
& = & (n-1)\lceil \log_2 n \rceil. &
\end{array}$$

□

As for *min-sort*, let us conclude with some experiments to check whether the number of comparisons between sequence elements is indeed a good indicator for the runtime in practice. The results in Table 8 look very different from the ones in Table 7.

Since `merge_sort` incurs much less comparisons than `minimum_sort`, our unit here is just **Mcomp**, the number of Megacomparisons (10^6 comparisons), according to Theorem 2. In other words, **Mcomp**$= 10^{-6}(n-1)\lceil \log_2 n \rceil$. **Time** is the absolute runtime of the program, this time in seconds and not minutes. But as in Table 7, **sec/Gcomp** tells us how many seconds the program needs to perform one Gigacomparison.

We first observe that this latter number *decreases* with n, where the rate of decrease becomes smaller and smaller. On our platform, we can go up to roughly $n = 51,200,000$ and find that **sec/Gcomp** continues like this in Table 8: 154, 146, 135, 128, 124.

n	100,000	200,000	400,000	800,000	1,600,000
Mcomp	1.7	3.6	7.6	16	33.6
Time (sec)	0.75	1.29	1.96	3.20	5.36
sec/Gcomp	441	358	257	200	160

Table 8: *Number of comparisons and runtime of* merge-sort

This seems to indicate that the runtime is proportional to the number of comparisons only for very large n. If you think about it, this is not surprising. Cheap operations that are performed n times, say, eat up a much higher fraction of the total runtime when n is small. This is because n is relatively large compared to the upper bound of $(n-1)\lceil\log_2 n\rceil$ on the number of comparisons between sequence elements. But since we ignore the cheap operations in our comparison count, this count is too optimistic for small n. Only as n becomes very large, the ratio between n and $(n-1)\lceil\log_2 n\rceil$ becomes negligible and we start to see the predicted proportionality.

For *minimum-sort*, this phenomenon does not show since n is negligible compared to $n(n-1)/2$ already for small n.

The most positive news of Table 8 is that `merge_sort` is actually a practical sorting algorithm. While it takes `minimum_sort` more than two hours to process $1,600,000$ numbers, `merge_sort` does the same in around 5 seconds. This is mainly due to the fact that $(n-1)\lceil\log_2 n\rceil$ is a much smaller number than $n(n-1)/2$, the number of comparisons needed by `minimum_sort` (that's why we switched from Gcomp to Mcomp).

On the other hand, the time needed by `merge_sort` per Gcomp is dramatically higher than in `minimum-sort`; for $n = 1,600,000$, we observe a factor of around 50. It may be surprising that the factor is this large, but the fact that it *is* larger can be explained. `merge_sort` is a more complicated algorithm than `minimum-sort`, with its recursive structure, the extra memory needed for the new deck, etc. The price to pay is that less comparisons can be done per second, since a lot of time is needed for other operations. But this is a moderate price, since we can more than pay for it by the gain in total runtime.

3.2.7 Lindenmayer systems

In this final section we want to present another application in which recursion is predominant and difficult to avoid (an iterative version would indeed require an explicit stack). As a bonus, this applications lets us draw beautiful pictures.

Let us first fix an *alphabet* Σ which is simply a finite set of symbols, for example $\Sigma = \{F, +, -\}$. Let Σ^* denote the set of all *words* that we can form from symbols in Σ. For example, $F + F+ \in \Sigma^*$.

Next, we fix a function $P : \Sigma \to \Sigma^*$. P maps every symbol to a word, and these are the *productions*. We might for example have the productions

Figure 20: *The turtle before and after processing the command sequence* $F + F+$

σ	$\mapsto$	$P(\sigma)$
F	$\mapsto$	$F + F+$
$+$	$\mapsto$	$+$
$-$	$\mapsto$	$-$

Finally, we fix an *initial word* $s \in \Sigma^*$, for example $s = F$.

The triple $\mathcal{L} = (\Sigma, P, s)$ is called a *Lindenmayer system*. Such a system generates an infinite sequence of words $s = w_0, w_1, \ldots$ as follows. To get the next word w_i from the previous word w_{i-1}, we simply substitute all symbols in w_{i-1} by their productions.

In our example, this yields

$$\begin{aligned}
w_0 &= F, \\
w_1 &= F + F + \\
w_2 &= F + F + +F + F + + \\
w_3 &= F + F + +F + F + + + F + F + +F + F + + + \\
&\vdots
\end{aligned}$$

The next step is to "draw" these words, and this gives the pictures we were talking about.

Turtle graphics. Imagine a turtle sitting at some point p on a large piece of paper, with its head pointing in some direction, see Figure 20 (left). The turtle can understand the commands F, $+$, and $-$. F means "move one step forward", $+$ means "turn counter-clockwise by an angle of 90 degrees", and $-$ means "turn clockwise by an angle of 90 degrees". The turtle can process any sequence of such commands, by executing them one after another. We are interested in the resulting path taken by the turtle on the piece of paper. The path generated by the command sequence $F + F+$, for example, is shown in Figure 20 (right), along with the position and orientation of the turtle after processing the command sequence.

The turtle can therefore graphically interpret any word generated by a Lindenmayer system over the alphabet $\{F, +, -\}$.

Recursively drawing Lindenmayer systems. For $\sigma \in \Sigma$, let w_i^σ denote the word resulting from σ by the i-fold substitution of all symbols according to their productions. In our running example, we have for example $w_2 = w_2^F = F + F + +F + F + +$ and $w_i^+ = +$ for all i.

The point is now that can we express w_i^σ in terms of the w_{i-1}'s of other symbols, and this is where recursion comes into play. Suppose that $P(\sigma) = \sigma_1 \cdots \sigma_k$. Then we can obtain w_i^σ as follows. We first substitute σ by $\sigma_1 \cdots \sigma_k$ (1-fold substitution), and in the resulting word $\sigma_1 \cdots \sigma_k$ we apply $(i-1)$-fold substitution to all the symbols. This shows that $=$

$$w_i^\sigma = w_{i-1}^{\sigma_1} \cdots w_{i-1}^{\sigma_k}.$$

This formula also implies that the drawing of w_i^σ is obtained by simply concatenating the drawings for $w_{i-1}^{\sigma_1}, \ldots, w_{i-1}^{\sigma_k}$. To get the actual word w_i, we simply concatenate the drawings of all w_i^σ, for σ running through the symbols of the initial word s.

Program 29 shows how this works for our running example with productions $F \mapsto F + F+, + \mapsto +, - \mapsto -$ and initial word F. Since $P^i(+) = +, P^i(-) = -$ for all i, we do not need to substitute $+$ and $-$ and get

$$w_i = w_i^F = w_{i-1}^F + w_{i-1}^F + . \tag{3.2}$$

The program assumes the existence of a library `turtle` with predefined turtle command functions `forward`, `left` (counterclockwise rotation with some angle) and `right` (clockwise rotations with some angle) in namespace `ifm`.

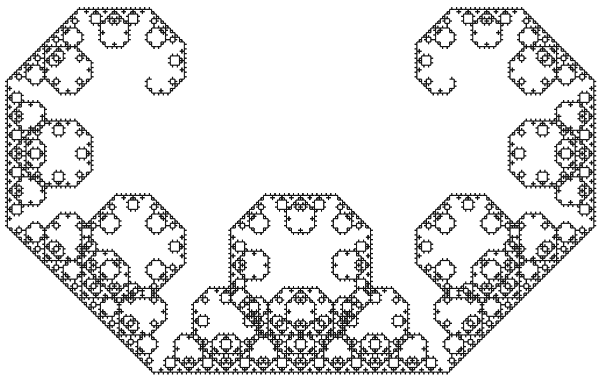
In the documentation of the program, we have omitted the "trivial" productions $+ \mapsto +, - \mapsto -$, and in specifying a Lindenmayer system, we can do so as well: we will usually only list productions for symbols that are not mapped to themselves.

```
// Prog: lindenmayer.C
// Draw turtle graphics for the Lindenmayer system with
// production F -> F+F+ and initial word F.

#include <iostream>
#include <IFM/turtle>

// POST: the word w_i^F is drawn
void f (unsigned int i) {
  if (i == 0)
    ifm::forward();   // F
  else {
    f(i-1);           // w_{i-1}^F
    ifm::left(90);    // +
    f(i-1);           // w_{i-1}^F
    ifm::left(90);    // +
  }
```

```
}

int main () {
  std::cout << "Number of iterations =? ";
  unsigned int n;
  std::cin >> n;

  // draw w_n = w_n(F)
  f(n);

  return 0;
}
```

Program 29: *progs/lindenmayer.C*

For input $n = 14$, the program will produce the following drawing.

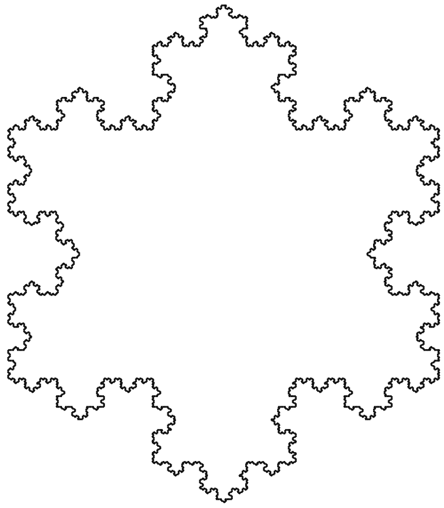


As n gets larger, the picture does not seem to change much; it rotates, and some more details develop, but apart from that the impression is the same. Assume you could draw the picture for $n = \infty$. Then equation (3.2) would give

$$w_\infty = w_\infty + w_\infty + .$$

This is a *self-similarity*: the drawing of w_∞ consists of two rotated drawings of itself. We have a *fractal*!

Additional features. We can extend the definition of a Lindenmayer system to include a rotation angle α that may be different from 90 degrees. This is shown in Program 30 that draws a snowflake for input $n = 5$.



```
// Prog: snowflake.C
// Draw turtle graphics for the Lindenmayer system with
// production F -> F-F++F-F, initial word F++F++F and
// rotation angle 60 degrees.
#include <iostream>
#include <IFM/turtle>

// POST: the word w_i^F is drawn
void f (unsigned int i) {
  if (i == 0)
    ifm::forward();  // F
  else {
    f(i-1);          // w_{i-1}^F
    ifm::right(60);  // -
    f(i-1);          // w_{i-1}^F
    ifm::left(120);  // ++
    f(i-1);          // w_{i-1}^F
    ifm::right(60);  // -
    f(i-1);          // w_{i-1}^F
  }
}

int main () {
  std::cout << "Number of iterations =? ";
  unsigned int n;
  std::cin >> n;

  // draw w_n = w_n^F++w_n^F++w_n^F
  f(n);              // w_n^F
```

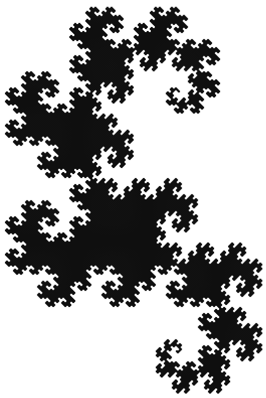
```
  ifm::left(120);     // ++
  f(n);               // w_n^F
  ifm::left(120);     // ++
  f(n);               // w_n^F

  return 0;
}
```

Program 30: *progs/snowflake.C*

To get more flexibility, we can also extend the alphabet Σ of symbols. For example, we may add symbols without any graphical interpretation; these are still useful, though, since they may be used in productions. For example, the Lindenmayer system with $\Sigma = \{F, +, -, X, Y\}$, initial word X and productions

$$\begin{array}{rcl} X & \mapsto & X + YF + \\ Y & \mapsto & -FX - Y \end{array}$$

yields the *dragon curve* (w_{14}, angle of 90 degrees).



The corresponding code is shown in Program 31.

```
// Prog: dragon.C
// Draw turtle graphics for the Lindenmayer system with
// productions X -> X+YF+, Y -> -FX-Y, initial word X
// and rotation angle 90 degrees
#include <iostream>
#include <IFM/turtle>

void y (unsigned int i);  // necessary: x and y call each other

// POST: w_i^X is drawn
void x (unsigned int i) {
```

```
  if (i > 0) {
    x(i-1);             // w_{i-1}^X
    ifm::left(90);      // +
    y(i-1);             // w_{i-1}^Y
    ifm::forward();     // F
    ifm::left(90);      // +
  }
}

// POST: w_i^Y is drawn
void y (unsigned int i) {
  if (i > 0) {
    ifm::right(90);     // -
    ifm::forward();     // F
    x(i-1);             // w_{i-1}^X
    ifm::right(90);     // -
    y(i-1);             // w_{i-1}^Y
  }
}

int main () {
  std::cout << "Number of iterations =? ";
  unsigned int n;
  std::cin >> n;

  // draw w_n = w_n^X
  x(n);

  return 0;
}
```

Program 31: *progs/dragon.C*

Finally, one can add symbols with graphical interpretation. Commonly used symbols are f (jump one step forward, this doesn't leave a trace), [(remember current position) and] (jump back to last remembered position). It is also typical to add new symbols with the same interpretation as F, say.

3.2.8 Details

Lindenmayer systems. Lindenmayer systems are named after the Danish biologist Aristide Lindenmayer (1925–1985) who proposed them in 1968 to model the growth of plants. Lindenmayer systems (with generalizations to 3-dimensional space) have found many applications in computer graphics.

3.2.9 Goals

Dispositional. At this point, you should ...

1) understand the concept of recursion, and why it makes sense to define a function through itself;
2) understand the semantics of recursive function calls and be aware that they do not always terminate;
3) appreciate the power of recursion in sorting and drawing Lindenmayer systems.

Operational. In particular, you should be able to ...

(G1) find pre- and postconditions for given recursive functions;

(G2) prove or disprove termination and correctness of recursive function calls;

(G3) translate recursive mathematical function definitions into C++ function definitions;

(G4) rewrite a given recursive function in iterative form;

(G5) recognize inefficient recursive functions and improve their performance;

(G6) count the number of operations of a given type in a recursive function call, using induction as the main tool;

(G7) write recursive functions for given tasks.

3.2.10 Exercises

Exercise 95 *Find pre- and postconditions for the following recursive functions.* (G1)

a)
```
bool f (int n)
{
  if (n == 0) return false;
  return !f(n-1);
}
```

b)
```
void g (unsigned int n)
{
  if (n == 0) {
    std::cout << "*";
    return;
  }
  g(n-1);
  g(n-1);
}
```

c)
```
unsigned int h (unsigned int n, unsigned int b) {
  if (n == 1) return 0;
  return 1 + h (n / b, b);
}
```

Exercise 96 *Prove or disprove for any of the following recursive functions that it terminates for all possible arguments. In this theory exercise, overflow should not be taken into account, i.e. you should pretend that the value range of* `unsigned int` *is equal to* $\mathbb{N}$. (G2)

a)
```
unsigned int f (unsigned int n)
{
  if (n == 0) return 1;
  return f(f(n-1));
}
```

b)
```
// POST: return value is the Ackermann function value A(m,n)
unsigned int A (unsigned int m, unsigned int n) {
  if (m == 0) return n+1;
  if (n == 0) return A(m-1,1);
  return A(m-1, A(m, n-1));
}
```

c)
```
unsigned int f (unsigned int n, unsigned int m)
{
  if (n == 0) return 0;
  return 1 + f ((n + m) / 2, 2 * m);
}
```

Exercise 97 *Consider the following recursive function defined on all nonnegative integers, also known as* McCarthy's 91 Function.

$$M(n) :== \begin{cases} n - 10, & \textit{if } n > 100 \\ M(M(n+11)), & \textit{if} n \leq 100. \end{cases}$$

a) *Provide a C++ function* `mccarthy` *that implements McCarthy's 91 Function.*

b) *What are the values of the following four function calls?*

(i) `mccarthy(101)`

(iii) `mccarthy(100)`

(iii) `mccarthy(99)`

(iv) `mccarthy(91)`

c) *Explain why the function is called McCarthy's 91 Function! More precisely, what is the value of* $M(n)$ *for any given number* n*?*

(G3)(G7)

Exercise 98

a) *Write and test a C++ function that computes binomial coefficients* $\binom{n}{k}$, $n, k \in \mathbb{N}$. *These may be defined in various equivalent ways. For example,*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!},$$

or

$$\binom{n}{k} := \begin{cases} 0, & \text{if } n < k \\ 1, & \text{if } n = k \text{ or } k = 0 \\ \binom{n-1}{k} + \binom{n-1}{k-1}, & \text{if } n > k, k > 0 \end{cases},$$

or

$$\binom{n}{k} := \begin{cases} 0, & \text{if } n < k \\ 1, & \text{if } n \geq k, k = 0 \\ \frac{n}{k}\binom{n-1}{k-1} & \text{if } n \geq k, k > 0 \end{cases}$$

b) *Which of the three variants is best suited for the implementation, and why? Argue theoretically, but underpin your arguments by comparing at least two different implementations of the function.*

(G3)(G5) (G7)

Exercise 99 *In how many ways can you own CHF 1? Despite its somewhat philosophical appearance, the question is a mathematical one. Given some amount of money, in how many ways can you partition it using the available denominations (bank notes and coins)? The denominations in CHF are 1000, 200, 100, 50, 20, 10 (banknotes), 5, 2, 1, 0.50, 0.20, 0.10, 0.05 (coins). The amount of CHF 0.20, for example, can be owned in four ways (to get integers, let's switch to centimes):* $(20), (10, 10), (10, 5, 5), (5, 5, 5, 5)$.

Solve the problem for any given input amount, by writing a program partition *that defines the following function (all values to be understood as centimes).*

```
// PRE: [first, last) is a valid nonempty range that describes
//      a sequence of denominations d_1 > d_2 > ... > d_n > 0
// POST: return value is the number of ways to partition amount
//       using denominations from d_1, ..., d_n
unsigned int partitions (unsigned int amount,
                         unsigned int* first,
                         unsigned int* last);
```

Use your program to determine in how many ways you can own CHF 1, and CHF 10. Can your program compute the number of ways for CHF 50? (G7)

Exercise 100 *Suppose you want to crack somebody's secret code, consisting of* d *digits between* 1 *and* 9*. You have somehow found out that exactly* k *of these digits are* 1*'s.*

a) *Write a program that generates all possible codes. The program should contain a function that solves the problem for given arguments* d *and* k*.*

b) *Adapt the program so that it also outputs the number of possible codes.*

For example, if $d = 2$ *and* $k = 1$*, the output may look like this:*

```
12 13 14 15 16 17 18 19 21 31 41 51 61 71 81 91
There were 16 possible codes.
```

(G7)

Exercise 101 *Rewrite the following recursive function in iterative form and test with a program whether your iterative version is correct. What can you say about the runtimes of both variants for values of* n *up to* 100*, say?* (G4)(G5)

```
unsigned int f (unsigned int n)
{
  if (n <= 2) return 1;
  return f(n-1) + 2 * f(n-3);
}
```

Exercise 102 *The following function finds an element with a given value* x *in a sorted sequence (if there is such an element).*

```
// PRE: [first, last) is a valid range, and the elements *p,
//      p in [first, last) are in ascending order
// POST: return value is a pointer p in [first, last) such
//       that *p = x, or the pointer last, if no such pointer
//       exists
int* binary_search (int* first, int* last, int x)
{
  int n = last - first;
  if (n == 0) return last;          // empty range
  if (n == 1)
    if (*first == x)
      return first;
    else
      return last;
  // n >= 2
  int* middle = first + n/2;
```

```
  if (*middle > x) {
    // x can't be in [middle, last)
    int* p = binary_search (first, middle, x);
    if (p == middle)
      return last; // x not found
    else
      return p;
  } else
    // *middle <= x; we may skip [first, middle)
    return binary_search (middle, last, x);
}
```

What is the maximum number $T(n)$ *of comparisons between sequence elements and* x *that this function performs if the number of sequence elements is* n*? Try to find an upper bound on* $T(n)$ *that is as good as possible. (You may use the statement of Exercise 103.)* (G6)

Exercise 103 *For any natural number* $n \geq 2$*, prove the following two (in)equalities.* (G6)

$$\lceil \log_2 \lfloor \frac{n}{2} \rfloor \rceil \leq \lceil \log_2 \lceil \frac{n}{2} \rceil \rceil = \lceil \log_2 n \rceil - 1.$$

Exercise 104 *Write programs that produce turtle graphics drawings for the following Lindenmayer systems* (Σ, P, s). (G7)

a) $\Sigma = \{F, +, -\}$, $s = F + F + F + F$ and P given by

$$F \mapsto FF + F + F + F + F + F - F.$$

b) $\Sigma = \{X, Y, +, -\}$, $s = Y$, and P given by

$$\begin{aligned} X &\mapsto Y + X + Y \\ Y &\mapsto X - Y - X. \end{aligned}$$

For the drawing, use rotation angle $\alpha = 60$ degrees and interpret *both* X and Y as "move one step forward".

c) Like b), but with the productions

$$\begin{aligned} X &\mapsto X + Y + +Y - X - -XX - Y + \\ Y &\mapsto -X + YY + +Y + X - -X - Y. \end{aligned}$$

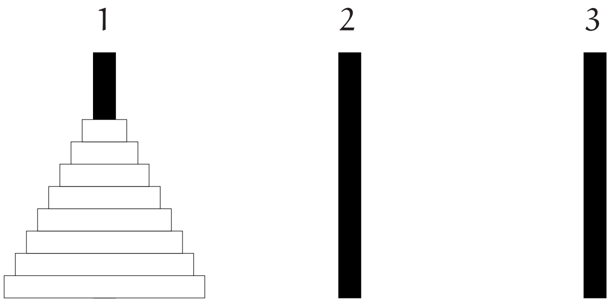


Figure 21: *The Tower of Hanoi*

Exercise 105 *The* Towers of Hanoi *puzzle (that can actually be bought from shops) is the following. There are three wooden pegs labeled* $1, 2, 3$*, where the first peg holds a stack of* n *disks, stacked in decreasing order of size, see Figure Figure 21.*

The goal is to transfer the stack of disks to peg 3, by moving one disk at a time from one peg to another. The rule is that at no time, a larger disk may be on top of a smaller one. For example, we could start by moving the topmost disk to peg 2 *(move* $(1,2)$*), then move the next disk from peg* 1 *to peg* 3 *(move* $(1,3)$*), then move the smaller disk from peg* 2 *onto the larger disk on peg* 3 *(move* $(2,3)$*), etc.*

Write a program `hanoi.C` *that outputs a sequence of moves that does the required transfer, for given input* n*. For example, if* $n = 2$*, the above initial sequence* $(1,2)(1,3)(2,3)$ *is already complete and solves the puzzle. Check the correctness of your program by hand at least for* $n = 3$*, by manually reproducing the sequence of moves on a piece of paper (or an actual Tower of Hanoi, if you have one).* (G7)

3.2.11 Challenges

Exercise 106 *On the occasion of major sports events, the Italian company Panini sells stickers to be collected in an album. For the EURO 2008 soccer championship, the collection comprised of 555 different stickers, available in packages of five stickers each.*

When buying a package, you cannot see which stickers it contains. The company only guarantees that each package contains five different stickers. Let us assume that each possible selection of five different stickers is equally likely to be contained in any given package. How many packages do you need to buy on average in order to have all the stickers?

For the case of EURO 2008 with 555 stickers, a newspaper claimed (based on consulting a math professor) that this number is 763. How did the professor arrive at that number, and is it correct?

Write a program that computes the average number of packages that you need to buy for a collection of size n*. (As a simple check, you should get one package on average if* $n = 5$*). What do you get for* $n = 555$*?*

Note: *In order to solve this challenge in a mathematically sound way, you need some basic knowledge of probability theory. But for our purposes, it is also ok to just handwave why your program is correct.*

Exercise 107 *Lindenmayer systems can also be used to draw (quite realistic) plants, with the growth process simulated by the various iterations. For this, however, there must be a possibility of creating branches. Let us therefore enhance our default set* $\{\mathsf{F}, +, -\}$ *of symbols with fixed meaning and now use* $\Sigma = \{\mathsf{F}, +, -, [,], \mathsf{f}\}$. *The symbol* [*is defined to have the following effect: the current state of the turtle (position and direction) is put on top of the* state stack *which is initially empty. The symbol*] *sets the state of the turtle back to the one found on top of the state stack, and removes the top state from the stack. This mechanism can be used to remember a certain state and return to it later.*

For example, if the rotation angle is 45 *degrees, the word* FF[+F][−F] *produces the drawing of Figure 22.*

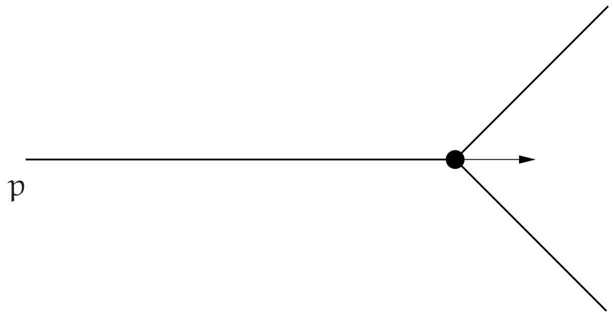


Figure 22: *The turtle after processing the command sequence* FF[+F][−F]

This does not look like a very sophisticated plant yet, but if you for example try the production

$$\mathsf{F} \quad \mapsto \quad \mathsf{FF}+[+\mathsf{F}-\mathsf{F}-\mathsf{F}]-[-\mathsf{F}+\mathsf{F}+\mathsf{F}]$$

with initial word F, *rotation angle* 22 *degrees, and four iterations, you will see what we mean.*

It remains to explain what the symbol f *means. It has the same effect on the state of the turtle as* F, *except that it does not draw a line. You can imagine that* f *makes the turtle "jump".*

Here are the functions of the library `turtle` *that correspond to this additional functionality.* `jump` *realizes* f, *while* `save` *and* `restore` *are for* [*and*]. *In order to draw Figure 22, we can therefore use the following statements.*

```
ifm::forward(2);
ifm::save();
ifm::left(45);
```

```
ifm::forward();
ifm::restore();
ifm::save();
ifm::right(45);
ifm::forward();
ifm::restore();
```

Here you see that we can provide an integer to `forward` *telling it how many steps we want to move forward (the default that we always used before is* 1*).*

Now here comes the challenge: write a turtle graphics program `amazing.C` *that will knock our socks off! In other words, we are asking for the most beautiful / realistic / whatever picture that you can produce using the recursive drawing scheme on top of the turtle graphics commands introduced so far (there are still more commands that are more or less common, but our turtle library stops at* $\Sigma = \{\mathsf{F}, +, -, [,], \mathsf{f}\}$*).*

If you think that you can submit a crappy program and still earn full points, you're right. But we count on your sportsmanship to give your best!

Chapter 4

Compound Types

4.1 Structs, or Plain Old Data

A POD-struct is an aggregate class that has no nonstatic data members of type pointer to member, non-POD struct, non-POD union (or array of such types) or reference, and has no user-defined copy-assignment operator and no user-defined destructor.

Section 9, paragraph 4, of the ISO/IEC Standard 14882 (C++ Standard)

In this section, we show how structs *are used to group data and to obtain new types with application-specific functionality. You will also see how* operator overloading *can help in making new types easy and intuitive to use.*

Suppose we want to use *rational numbers* in a program, i.e., numbers of the form n/d, where both the numerator n and the denominator d are integers. C++ does not have a fundamental type for rational numbers, so we have to implement it ourselves.

We could of course represent a rational number simply by two values of type int, but this would not be in line with our perception of the rational numbers as a distinct mathematical concept. The two numbers n and d "belong together", and this is also reflected in mathematical notation: the symbol $\mathbb{Q}$ for the set of rational numbers indicates that we are dealing with a mathematical type, defined by its value range *and* its functionality (see Section 2.1.6). Ideally, we would like to get a C++ type that can be used like existing arithmetic types; the following piece of code (for adding two rational numbers) shows how this could look like.

```
// input
std::cout << "Rational number r:\n";
rational r;
std::cin >> r;

std::cout << "Rational number s:\n";
rational s;
std::cin >> s;

// computation and output
std::cout << "Sum is " << r + s << ".\n";
```

C++ offers several concepts for defining new types based on existing types. In this section, we introduce the concept of *structs*. A struct is used to aggregate several values of different types into one value of a new type. With this, we can easily model the mathematical type $\mathbb{Q}$ as a new type in C++. Here is a working program that makes a first step toward the desired piece of code above.

```
// Program: userational.C
// Add two rational numbers.
#include <iostream>

// the new type rational
struct rational {
  int n;
  int d; // INV: d != 0
};

// POST: return value is the sum of a and b
rational add (rational a, rational b)
{
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}

int main ()
{
  // input
  std::cout << "Rational number r:\n";
  rational r;
  std::cout << " numerator =?   "; std::cin >> r.n;
  std::cout << " denominator =? "; std::cin >> r.d;

  std::cout << "Rational number s:\n";
  rational s;
  std::cout << " numerator =?   "; std::cin >> s.n;
  std::cout << " denominator =? "; std::cin >> s.d;

  // computation
  rational t = add (r, s);

  // output
  std::cout << "Sum is " << t.n << "/" << t.d << ".\n";

  return 0;
}
```

Program 32: *progs/userational.C*

In C++, a struct defines a new type whose value range is the *Cartesian product* of a fixed number of types.[1] In our case, we define a new type named `rational` whose value range is the Cartesian product `int`$\times$`int`, where we interpret a value (n, d) as the quotient n/d.

Since there is no type for the denominator with the appropriate value range `int`$\setminus\{0\}$, we specify the requirement $d \neq 0$ by an informal *invariant*, a condition that has to hold for all legal combinations of values. Such an invariant is indicated by a comment starting with

```
// INV:
```

Like pre- and postconditions of functions (see Section 3.1.1), invariants are an informal way of documenting the program; they are not standardized, and our way of writing them is one possible convention.

The type `rational` is referred to as a *struct*, and it can be used like any other type; for example, it may appear as parameter type and return type in functions like `add`.

A struct defines a type, not variables. Let's get rid of one possible confusion right from the beginning. The definition

```
struct rational {
  int n;
  int d; // INV: d != 0
};
```

does *not* define variables `n` and `d` of type `int`, although the two middle lines look like variable declarations as we know them. Rather, all four lines together define a *type* of the name `rational`, but at that point, neither a variable of that new type, nor variables of type `int` have been defined. The two middle lines

```
  int n;
  int d; // INV: d != 0
```

specify that any actual *object* of the new type (i.e. any concrete rational number) "has" (is represented by) two objects of type `int` that can be accessed through the names `n` and `d`; see the member access below. This specification is important if we want to implement operations on our new type like in the function `add`.

Here is an analogy for the situation. If the university administration wants to specify how a student is represented in their files, they might come up with three pieces of data that are necessary: a name, an identification number, and a program of study. This defines the "type" of a student and allows functionality (registration, change of program of study, etc.) to be realized, long before any students actually show up.

[1]Here and in the following, we identify a type with its value range to avoid clumsy formulations.

4.1.1 Struct definitions.

In general, a struct definition looks as follows.

```
struct T {
  T1 name1;
  T2 name2;
  ...
  TN nameN;
};
```

Here, T is the name of the newly introduced struct (this name must be an identifier, Section 2.1.9), and $T1,\ldots,TN$ are names of existing types. These are called the *underlying types* of T. The identifiers *name1*, *name2*,..., *nameN* are are the *data members* of the new type T.

The value range of T is $T1 \times T2 \times \ldots \times TN$. This means, a value of type T is an N-tuple $(t_1, t_2, \ldots, t_N)$ where $t_i \in Ti$.

"Existing types" might be fundamental types, but also user-defined types. For example, consider the vector space $\mathbb{Q}^3$ over the field $\mathbb{Q}$. Given the type `rational` as above, we could model $\mathbb{Q}^3$ as follows.

```
struct rational_vector_3 {
  rational x;
  rational y;
  rational z;
};
```

Although it follows from the definition, let us make it explicit: the types $T1,\ldots,TN$ need not be the same. Here is an example: If $0, 1, \ldots, U$ is the value range of the type `unsigned int`, we can get a variant of the type `int` with value range

$$\{-U, -U+1, \ldots, -1, 0, 1, \ldots, U-1, U\}$$

as follows.

```
struct extended_int {
  // represents u if n==false and -u otherwise
  unsigned int u;  // absolute value
  bool         n;  // sign bit
};
```

The value range of this type is $\{0, 1, \ldots, U\} \times \{true, false\}$, but like in the rational case, we interpret values differently: a value (u, n) "means" u if $n = false$ and $-u$ if $n = true$.

Even if two struct definitions have the same *member specification* (the part of the definition enclosed in curly braces), they define *different* types, and it is not possible to replace one for the other. Consider this trivial but instructive example with two apparently equal structs defined over an empty set of existing types.

```
struct S {
};

struct T {
};

void foo (S s) {}

int main() {
  S s;
  T t;
  foo (s); // ok
  foo (t); // error: type mismatch
  return 0;
}
```

It is also possible to use array members in structs. For example, the field $\mathbb{Q}^3$ that we have discussed above could alternatively be modeled like this.

```
struct rational_vector_3 {
  rational v[3];
};
```

4.1.2 Structs and scope

The scope of a struct is the part of the program in which it can be used (in a variable declaration, or as a formal function parameter type, for example). Structs behave similar to functions here: the scope of a struct is the union of the scopes of all its *declarations*, where a struct declaration has the form

struct T

The struct definition is a declaration as well, and usually one actually needs the definition in order to use a struct. This is easy to explain: in order to translate a variable declaration of struct type, or a function with formal parameters of struct type into machine language, the compiler needs to know the amount of memory required by an object of the struct. But this information is only obtainable from the definition of the struct; as long as the compiler has only seen a declaration of T, the struct T is said to have *incomplete type*.

4.1.3 Member access

A struct is more than the Cartesian product of its underlying types—it offers some basic functionality on its own that we explain next. The most important (and also most visible) functionality of a struct is the access to the data members (the values t_i in the

N-tuple $t = (t_1, \ldots, t_N)$), and here is where the identifiers *name1*,..., *nameN* come in. If *expr* is an expression of type T with value $(t_1, \ldots, t_N)$, then t_K—the K-th component of its value—can be accessed as

expr.nameK

Here, '`.`' is the *member access operator* (see Table 9 in the Appendix for its specifics). The composite expression *expr* `.` *nameK* is an lvalue if *expr* itself is an lvalue, and we say that the data member *nameK* is *accessed for expr*.

Lines 25 and 26 of Program 32 assign values to the rational numbers `r` through the member access operator, while line 37 employs the member access operator to output the value of the rational number `t`. The additional output of '/' indicates that we interpret the 2-tuple (n, d) as the quotient n/d.

4.1.4 Initialization and assignment

We can initialize objects of struct type and assign values to them, just like we do it for fundamental types.

In line 34 of Program 32 for example, the variable `t` of type `rational` is initialized with the value of the expression `add (r, s)`. In a struct, initialization is quite naturally done member-wise, i.e. for each data member separately. Under the hood, the declaration statement

```
rational t = add (r, s);
```

therefore has the effect of initializing `t.n` (with the first component of the value of `add (r, s)`) and `t.d` (with the second component). Interestingly, this also works with array members. Structs therefore provide a way of "faking" array initialization and assignment by wrapping the array into a struct. Here is an example to show what we mean.

```
#include<iostream>

struct point {
  double coord[2];
};

int main()
{
  point p;
  p.coord[0] = 1;
  p.coord[1] = 2;

  point q = p;
  std::cout << q.coord[0] << " "     // 1
```

```
            << q.coord[1] << "\n";  // 2

  return 0;
}
```

This works since the data members of a struct object occupy a contiguous part of the main memory, and since (in contrast to array types) struct types "know" their memory requirements. From this, the compiler can figure out how many memory cells are to be copied for the initialization of q in `point q = p`.

In the same way (memberwise initialization), the formal parameters `a` and `b` of the function `add` are initialized from the values of `r` and `s`; the value of `add (r, s)` itself also results from an initialization of a (temporary) object when the `return` statement of the function `add` is executed.

Instead of the above declaration statement that initializes `t` we could also have written

```
rational t;
t = add (r, s);
```

Here, `t` is *default-initialized* first, and this default-initializes the data members. In our case, they are of type `int`; for fundamental types, default-initialization does nothing, so the values of the data members are undefined after default-initialization (see also Section 2.1.8). In the next line, the value of `add (r, s)` is assigned to `t`, and this assignment again happens member-wise.

What about other operations? For every fundamental type T, two expressions of type T can be tested for equality, using the operators `==` and `!=`. It would therefore seem natural to have these two operators also for structs, implemented in such a way that they compare member-wise.

Formally, this would be correct: if $t = (t_1, \ldots, t_N)$ and $t' = (t'_1, \ldots t'_N)$, then we have $t = t'$ if and only if $t_K = t'_K$ for $K = 1, \ldots, N$.

But our type `rational` already shows that this won't work: under member-wise equality, we would erroneously conclude that $2/3 \neq 4/6$. The problem is that the *syntactical value range* `int`×`int` of the type `rational` does not coincide with the *semantical value range* in which we identify pairs (n, d) that define the same rational number n/d.

The same happens with our type `extended_int` from above: since both pairs $(0, \mathit{false})$ and $(0, \mathit{true})$ are interpreted as 0, member-wise equality would give us "$0 \neq 0$" in this case.

Only the implementor of a struct knows the semantical value range, and for this reason, C++ neither provides equality operators for structs, nor any other operations beyond the member access, initialization, and assignment discussed above. Operations that respect the semantical value range can be provided by the implementor, though, see next section.

You might argue that even member-wise initialization and assignment could be inconsistent with the semantics of the type. Later, we will indeed encounter such a situation, and we will show how it can be dealt with elegantly.

4.1.5 User-defined operators

New types require new operations, but when it comes to the naming of such operations, one less nice aspect of Program 32 shows in line 34. By defining the function add, we were able to perform the operation $t := r + s$ through the statement

```
rational t = add (r, s);
```

Ideally, however, we would like to add rational numbers like we add integers or floating-point numbers, by simply writing (in our case)

```
rational t = r + s;
```

The benefit of this might not be immediately obvious, in particular since the naming of the function add seems to be quite reasonable; but consider the expression

```
rational t = subtract (multiply (p, q), multiply (r, s));
```

and its "natural" counterpart

```
rational t = p * q - r * s;
```

to get an idea what we mean.

The natural notation can indeed be achieved: a key feature of the C++ language is that most of its operators (see Table 9 in the Appendix for the full list) can be *overloaded* to work for other types as well. This means that we can use the same operator token to implement various operators: we "overload" the token.

In principle, this is nothing new: we already know that the binary operator + is available for several types, for example int and double. What is new is that we can add even more overloads on our own, and simply let the compiler figure out from the call parameter types which one is needed in a certain context.

In overloading an operator, we cannot change the operator's arity, precedence or associativity, but we can create versions of it with arbitrary formal parameter and return types.

Operator overloading is simply a special case of *function overloading*. For example, having the structs rational and extended_int available, we could declare the following two functions in the same program, without creating a name clash: for any call to the function square in the program, the compiler can find out from the call parameter type which of the two functions we mean.

```
// POST: returns a * a
rational square (rational a);

// POST: returns a * a
extended_int square (extended_int a);
```

Function overloading in general is useful, but not nearly as useful as operator overloading. To define an overloaded operator, we have to use the *functional operator notation*. In this notation, the name of the operator is obtained by appending its token to the prefix `operator`. In case of the binary addition operator for the type `rational`, this looks as follows and replaces the function `add`.

```
// POST: return value is the sum of a and b
rational operator+ (rational a, rational b)
{
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}
```

In Program 32, we can now replace line 34 by

```
rational t = r + s; // equivalent to rational t = operator+ (r, s);
```

Here, the comment refers to the fact that an operator can also be *called* in functional notation; in contrast, it appears in *infix notation* in `r + s`. The call in functional notation can be useful for didactic purposes, since it emphasizes the fact that an operator is simply a special function; in an application, however, the point is to avoid functional notation and use the infix notation.

The other three basic arithmetic operations are similar, and here we only give their declarations.

```
// POST: return value is the difference of a and b
rational operator- (rational a, rational b);

// POST: return value is the product of a and b
rational operator* (rational a, rational b);

// POST: return value is the quotient of a and b
// PRE:  b != 0
rational operator/ (rational a, rational b);
```

We can also overload the unary `-` operator; in functional operator notation, it has the same name as the binary version, but it has only one instead of two parameters. In the following implementation, we use the (modified) "local copy" of the call parameter a as the return value.

```
// POST: return value is -a
rational operator- (rational a)
{
  a.n = -a.n;
  return a;
}
```

In order to compare rational numbers, we need the relational operators as well. Here is the equality operator as an example.

```
// POST: return value is true if and only if a == b
bool operator== (rational a, rational b)
{
  return a.n * b.d == a.d * b.n;
}
```

4.1.6 Details

Overloading resolution. If there are several functions or operators of the same name in a program, the compiler has to figure out which one is meant in a certain function call. This process is called *overloading resolution* and only depends on the types of the call parameters. Overloading resolution is therefore done at compile time. There are two cases that we need to consider: we can either have an *unqualified* function call (like `add (r, s)` in Program 32), or a *qualified* function call (like `std::sqrt(2.0)`). To process an unqualified function call of the form

fname (*expression1*, ..., *expressionN*)

the compiler has to find a matching function declaration. Candidates are all functions `f` of name *fname* such that the function call is in the scope of some declaration of `f`. In addition, the number of formal parameters must match the number of call parameters, and each call parameter must be of a type whose values can be converted to the corresponding formal parameter types.

In a qualified function call of the form

X::*fname* (*expression1*, ..., *expressionN*)

where X is a namespace, only this namespace is searched for candidates.

Argument-dependent name lookup (Koenig lookup). There is one special rule that sometimes makes the list of candidates larger. If some call parameter type of an unqualified function call is defined in a namespace X (for example the namespace `std`), then the compiler also searches for candidates in X. This is useful mainly for operators and allows them to be called unqualified in infix notation. The point of using operators in infix notation would be spoiled if we had to mention a namespace somewhere in the operator call.

Resolution: Finding the best match. For each candidate function and each call parameter, it is checked how well the call parameter type matches the corresponding formal parameter type. There are four quality levels, going from better to worse, given in the following list.

(1) Exact match. The types of the call parameter and the formal parameter are the same.

(2) Promotion match. There is a promotion from the call parameter type to the formal parameter type. We have seen some examples for promotions, like from `bool` to `int` and from `float` to `double`.

(3) Standard conversion match. There is a standard conversion from the call parameter type to the formal parameter type. We have seen that all fundamental arithmetic types can be converted into each other by standard conversions.

(4) User-defined conversion match. There is a user-defined conversion from the call parameter type to the formal parameter type. We will get to user-defined conversions only later in this book.

A function f is called *better* than g with respect to a parameter, if the match that f induces on that parameter is at least as good as the match induced by g. If the match is really better, f is called *strictly better* for the parameter.

A function f is called a *best match* if it is better than any other candidate g in all parameters, and strictly better than g in at least one parameter.

Under this definition, there is at most one best match, but it may happen that there is no best match, in which case the function call is *ambiguous*, and the compiler issues an error message.

Here is an example. Consider the two overloaded function declarations

```
void foo(double d);
void foo(unsigned int u);
```

In the code fragment

```
float f = 1.0f;
foo(f);
```

the first overload is chosen, since `float` can be promoted to `double`, but only standard-converted to `unsigned int`. In

```
int i = 1;
foo(i);
```

the call is ambiguous, since `int` can be standard-converted to both `double` and `unsigned int`.

4.1.7 Goals

Dispositional. At this point, you should ...

1) know how structs can be used to aggregate several different types into one new type;
2) understand the difference between the syntactical and semantical value range of a struct;
3) know that C++ functions and operators can be overloaded.

Operational. In particular, you should be able to ...

(G1) define structs whose semantical value ranges correspond to that of given mathematical sets;

(G2) provide definitions of functions and overloaded operators on structs, according to given functionality;

(G3) write programs that define and use structs according to given functionality.

4.1.8 Exercises

Exercise 108 *Define a type* Tribool *for three-valued logic; in three-valued logic, we have the truth values true, false, and unknown.*

For the type Tribool, *implement the logical operators*

```
// POST: returns x AND y
Tribool operator&& (Tribool x, Tribool y);

// POST: returns x OR y
Tribool operator|| (Tribool x, Tribool y);
```

where AND ($\wedge$) and OR ($\vee$) are defined according to the following two tables. (G1)(G2)

$\wedge$	*false*	*unknown*	*true*
false	*false*	*false*	*false*
unknown	*false*	*unknown*	*unknown*
true	*false*	*unknown*	*true*

$\vee$	*false*	*unknown*	*true*
false	*false*	*unknown*	*true*
unknown	*unknown*	*unknown*	*true*
true	*true*	*true*	*true*

Test your type by writing a program that outputs these truth tables in some format of your choice.

Exercise 109 *Define a type* Z_7 *for computing with integers modulo 7. Mathematically, this corresponds to the finite ring $\mathbb{Z}_7 = \mathbb{Z}/7\mathbb{Z}$ of residue classes modulo 7.*

For the type Z_7, *implement addition and subtraction operators*

```
// POST: return value is the sum of a and b
Z_7 operator+ (Z_7 a, Z_7 b);

// POST: return value is the difference of a and b
Z_7 operator- (Z_7 a, Z_7 b);
```

according to the following table (this table also defines subtraction: $x - y$ is the unique number $z \in \{0, \ldots, 6\}$ such that $x = y + z$). (G1)(G2)

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Exercise 110 *Provide definitions for the following binary arithmetic operators on the type* rational. (G2)(G3)

```
// POST: return value is the difference of a and b
rational operator- (rational a, rational b);

// POST: return value is the product of a and b
rational operator* (rational a, rational b);

// POST: return value is the quotient of a and b
// PRE:  b != 0
rational operator/ (rational a, rational b);
```

Exercise 111 *Provide definitions for the following binary relational operators on the type* rational. *In doing this, try to reuse operators that are already defined.* (G2)(G3)

```
// POST: return value is true if and only if a != b
bool operator!= (rational a, rational b);

// POST: return value is true if and only if a < b
bool operator< (rational a, rational b);

// POST: return value is true if and only if a <= b
bool operator<= (rational a, rational b);

// POST: return value is true if and only if a > b
bool operator> (rational a, rational b);

// POST: return value is true if and only if a >= b
bool operator>= (rational a, rational b);
```

Exercise 112 *Provide definitions for the following binary arithmetic operators on the type* extended_int *(Page 235), and test them in a program (for that it could be helpful to provide an output facility for the type* extended_int, *and a function that assigns to an* extended_int *value a value of type* int*). As in the previous exercise, try to reuse code.* (G2)(G3)

```
// POST: return value is the sum of a and b
extended_int operator+ (extended_int a, extended_int b);

// POST: return value is the difference of a and b
extended_int operator- (extended_int a, extended_int b);

// POST: return value is the product of a and b
extended_int operator* (extended_int a, extended_int b);

// POST: return value is -a
extended_int operator- (extended_int a);
```

Exercise 113 *Consider the following set of three functions.*

```
void foo(double, double)       { ... }  // function A
void foo(unsigned int, int)    { ... }  // function B
void foo(float, unsigned int) { ... }  // function C
```

For each of the following function calls, decide to which of the functions ($\mathrm{A}, \mathrm{B}, \mathrm{C}$*) it resolves to, or decide that the call is ambiguous. Explain your decisions! This exercise requires you to read the paragraph on overloading resolution in the Details section.*

a) `foo(1, 1)`

b) `foo(1u, 1.0f)`

c) `foo(1.0, 1)`

d) `foo(1, 1u)`

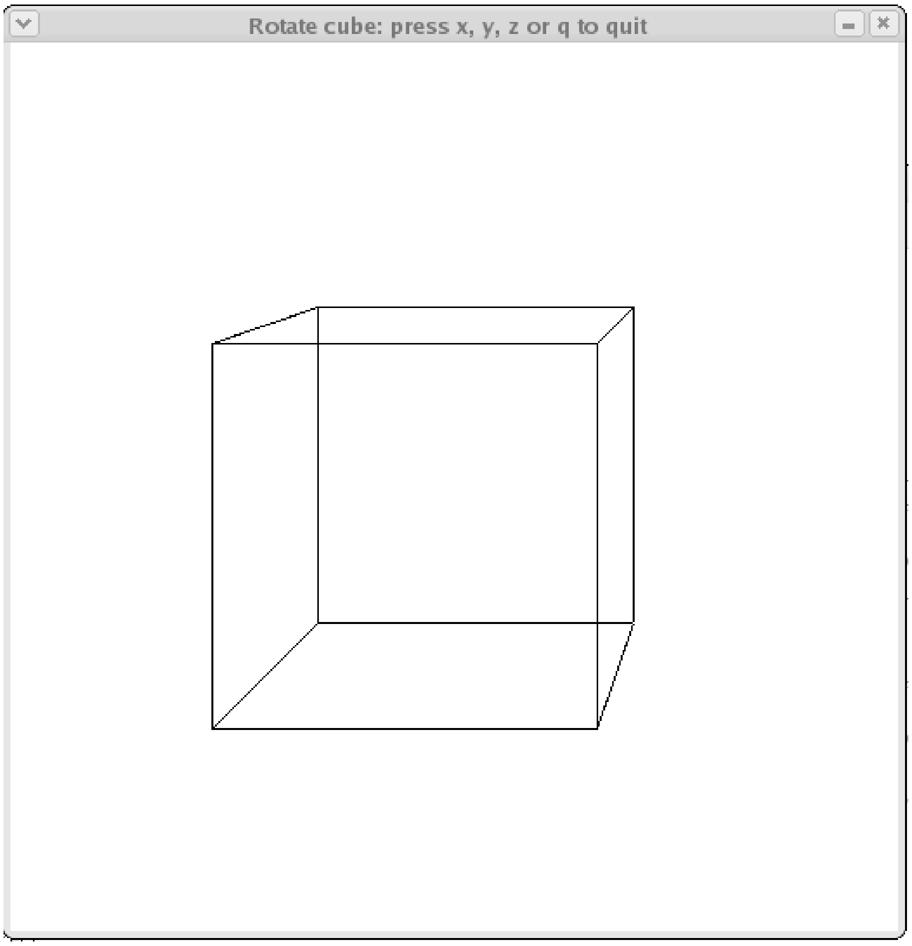
e) `foo(1, 1.0f)`

f) `foo(1.0f, 1.0)`

4.1.9 Challenges

Exercise 114 *This challenge has a computer graphics flavor. Write a program that allows you to visualize and manipulate a 3-dimensional object. For the sake of concreteness, think of a* wireframe model *of a cube given by 12 edges in threedimensional space.*

The program should be able to draw the object in perspective view and at least provide the user with a possibility of rotating the object around the three axes. The drawing window might for example look like this:



Instead of a cube, you may want to take another platonic solid, you may read the wireframe model from a file, you may add the possibility of scaling the object, translating it, etc. Use the library `libwindow` *that is available at the course homepage to create the graphical output.*

If you don't know (or have forgotten) how to rotate and project threedimensional points, here is a crash course.

Rotating a point $(x, y) \in \mathbb{R}^2$ *around the origin by an angle of* α *(radians) results in the point* (x', y') *with coordinates*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In order to rotate a point $(x, y, z) \in \mathbb{R}^3$ *around the z-axis, you simply keep z unchanged and rotate* (x, y) *as above. By symmetry, you can figure out how this works for the other axes.*

General perspective projection is not so easy, but if you want to project a point onto the $z = 0$*-plane (imagine that this plane is the computer screen that you want to draw on), this is not hard. Imagine that* $v = (v_x, v_y, v_z)$ *is the viewpoint* *(position of your eye).* $v_z > 0$ *for example means that you are sitting in front of the screen. When you project the point* $p = (x, y, z)$ *onto the screen, the image point has coordinates*

$$(x - t(v_x - x), y - t(v_y - y)),$$

where

$$t = \frac{z}{v_z - z}.$$

The projection thus only works if $v_z \neq z$.

4.2 Type Variants

Don't believe the type.

Anonymous

This section explains two ways of obtaining variants of a given type that have the same value range but differ in certain functionality aspects. Reference types *enable functions to accept and return lvalues and in particular change the values of their formal parameters.* Const-types *allow us to define values as being non-modifiable, in such a way that the compiler can detect illegal modifications. Reference types and const-types can be combined and naturally come up in implementing functionality for structs.*

4.2.1 Reference types

Let us now try to implement the addition assignment operator += for the struct `rational` from Program 35. Here is an attempt:

```
rational operator+= (rational a, rational b) {
  a.n = a.n * b.d + a.d * b.n;
  a.d *= b.d;
  return a;
}
```

With this, we can write

```
rational r;
r.n = 1; r.d = 2;                          // 1/2

rational s;
s.n = 1; s.d = 3;                          // 1/3

r += s;
std::cout << r.n << "/" << r.d << "\n";
```

You may already see that the output of this will not be the desired 5/6. Recall from Section 3.1.3 what happens when `r += s` (equivalently, `operator+= (r, s)`) is evaluated: `r` and `s` are evaluated, and the resulting values are used to initialize the formal parameters `a` and `b` of the function `operator+=`. The values of `r` and `s` are not changed by the function call.

Hence, with the above implementation of `operator+=`, the *value* of the expression `r += s` is indeed $5/6$, but the desired *effect*, the increment of `r`, does not happen. That's why we get 1/2 as output in the above piece of code.

In order to implement operator+= properly, we must enable functions to change the values of their call parameters. Surprisingly, we do not need a new concept for that on the function side; we simply need a new category of types.

Definition. If T is any type, then

$T\&$

is the corresponding *reference type* (read $T\&$ as "T reference" or "reference to T"). In value range and functionality, $T\&$ is identical to T. The difference is in the initialization and assignment semantics.

A variable of reference type $T\&$ (also called a *reference*) can be initialized only from an *lvalue* of type T, or any type whose values can be converted to T. The initialization makes it an *alias* of the lvalue: another name for the object behind the lvalue. We also say that the reference *refers* to that object. The following example shows this.

```
int i = 5;
int& j = i;              // j becomes an alias of i

j = 6;                   // changes the value of  i
std::cout << i << "\n";  // outputs 6
```

A reference cannot be changed to refer to another object after initialization. If we later assign something to the reference, we in fact assign to the *object* referred to by it. In writing `j = 6` in the above piece of code, we therefore change the value of `i` to 6, since `j` is an alias of `i`.

Internally, a value of type $T\&$ is represented by the address of the object it refers to. This explains why we need an lvalue to initialize a reference type variable, and why things like

```
int& j;       // error: j must be an alias of something
int& k = 5;   // error: the literal 5 has no address
```

don't work. Any expression of reference type is an lvalue itself. We can therefore use a reference `r` to initialize another reference `rr`, but then we don't get a reference to `r`, but another reference to the object referred to by `r`:

```
int i = 5;
int& j = i;  // j becomes an alias of i
int& k = j;  // k becomes another alias of i
```

4.2.2 Call by value and call by reference

When a function has a formal parameter of reference type, the corresponding call parameter must be an lvalue; when the function call is evaluated, the initialization of the formal parameter makes it an alias of the call parameter. In this way, we can implement functions that change the values of their call parameters. Here is an example.

```
void increment (int& i)
{
  ++i;
}

int main ()
{
  int j = 5;
  increment (j);
  std::cout << j << "\n"; // outputs 6

  return 0;
}
```

If a formal parameter of a function has reference type, we have *call-by-reference* semantics with respect to that parameter. Equivalently, we say that we *pass* the parameter by reference.

If the formal parameter is not of reference type, we have *call-by-value* semantics: we pass the parameter by value. Under call by reference, the address of (or a reference to) the call parameter is used to initialize the formal parameter; under call-by-value semantics, it is the value of the call parameter that is used for initialization.

The basic rule is to pass a parameter by reference only if the function in question actually needs to change the call parameter value. If that is not the case, call by value is more flexible, since it allows a larger class of call parameters (lvalues *and* rvalues instead of lvalues only).

4.2.3 Return by value and return by reference

The return type of a function can be a reference type as well, in which case we have *return-by-reference* semantics (otherwise, we *return by value*). If the function returns a reference, the function call expression is an lvalue itself, and we can use it wherever lvalues are expected.

This means that the function itself chooses (by using reference types or not) whether its call parameters and return value are lvalues or rvalues. Section 2.1.13 and Section 2.2.4 document these choices for some of the operators on fundamental types, but only now we understand the mechanism that makes such choices possible.

As a concrete example, let us consider the following version of the function `increment` that exactly models the behavior of the pre-increment operator ++: it increments its lvalue parameter and returns it as an lvalue.

```
int& increment (int& i)
{
```

```
  return ++i;
}
```

In general, we must make sure that an expression of reference type that we return refers to a *non-temporary* object. To understand what a temporary object is, let us consider the following function.

```
int& foo (int i)
{
  return i;
}
```

This is asking for trouble, since the formal parameter `i` runs out of scope when the function call terminates. This means that the associated memory is freed and the address expires (see Section 2.4.3). If we now write for example

```
int i = 3;
int& j = foo(i);         // j refers to expired object
std::cout << j << "\n"; // undefined behavior
```

the reference `j` refers to an expired object, and the resulting behavior of the program is undefined.

Reference Guideline: Whenever you create an alias for an object, ensure that the object does not expire before the alias.

The compiler usually notices violations of the Reference Guideline and issues a warning.

4.2.4 More user-defined operators

Rational numbers: addition assignment. Let's get back to the addition assignment operator for our new struct `rational`. In order to fix our failed attempt from the beginning of this section, we need to add two characters only.

As in the previous function `increment`, the formal parameter `a` must be passed as a reference, and to be compliant with the usual semantics of `+=`, we also return the result as a reference:

```
// POST: b has been added to a; return value is the new value of a
rational& operator+= (rational& a, rational b)
{
  a.n = a.n * b.d + a.d * b.n;
  a.d *= b.d;
  return a;
}
```

The other arithmetic assignment operators are similar, and we don't list them here explicitly. Together with the arithmetic and relational operators discussed in Section 4.1.5, we now have a useful set of operations on rational numbers.

Rational numbers: input and output. Let us look at Program 35 once more, with the function name add replaced by operator+ and the function call add (r, s) replaced by r + s. Still, we can spot potential improvements: instead of writing

```
std::cout << "Sum is " << t.n << "/" << t.d << "\n";
```

in line 37, we'd rather write

```
std::cout << "Sum is " << t << "\n";
```

just like we are doing it for fundamental types. After all, we want to think of a rational number as a single value from the set $\mathbb{Q}$ and not as two values from the set $\mathbb{Z}$.

From what we have done above, you can guess that all we have to do is to overload the output operator <<. In discussing the output operator in Section 2.1.13 we have argued that the output stream passed to and returned by the output operator must be an lvalue, since the output operator modifies the stream. Having reference types at our disposal, this can easily be done: we simply pass and return the output stream (whose type is std::ostream) as a reference:

```
// POST: a has been written to o
std::ostream& operator<< (std::ostream& o, rational r)
{
  return o << r.n << "/" << r.d;
}
```

There is no reason to stop here: for the input, we would in the same fashion like to replace the two input statements std::cin >> r.n; and std::cin >> r.d; by the single statement

```
std::cin >> r;
```

(and the same for the input of s). Again, we need to pass and return the input stream (of type std::istream) as a reference. In addition, we must pass the rational number that we want to read as a reference, since the input operator has to modify its value.

The operator first reads the numerator from the stream, followed by a separating character, and finally the denominator. Thus, we can read a rational number in one go by entering for example 1/2.

```
// POST: r has been read from i
// PRE:  i starts with a rational number of the form "n/d"
std::istream& operator>> (std::istream& i, rational& r)
{
  char c; // separating character, e.g. '/'
  return i >> r.n >> c >> r.d;
}
```

In contrast to operator<<, things can go wrong, e.g., if the user enters the character sequence "A/B" when prompted for a rational number. Also, we probably don't want to accept 3.4 as a rational number as our input operator does. There are mechanisms to deal with such issues, but we won't discuss them here.

Let us conclude this section with a beautified version of Program 35. What makes this version even nicer is the fact that in the `main` function, the new type is used exactly like an "atomic" fundamental type such as `int`.

In the spirit of Section 3.1.8 on modularization, we actually split the program into three files: a file `rational.h` that contains the definition of the struct `rational`, along with declarations of the overloaded operators; a file `rational.C` that contains the definitions of these operators; finally, a file `userational2.C` that contains the main program. At the same time, we put our new type `rational` and the operations on it into namespace `ifm` in order to avoid possible name clashes.[2] Exercise ?? asks you to actually integrate the new rational number type into the math library that you have built in Exercise 90, so that Program 33 below can be compiled using this library.

```
// Program: userational2.C
// Add two rational numbers.
#include <iostream>
#include "rational.C"

int main ()
{
  // input
  std::cout << "Rational number r:\n";
  rational r;
  std::cin >> r;

  std::cout << "Rational number s:\n";
  rational s;
  std::cin >> s;

  // computation and output
  std::cout << "Sum is " << r + s << ".\n";

  return 0;
}
```

Program 33: *progs/userational2.C*

```
// Program: rational.h
// Define a type for rational numbers, and declare
// operations on it.
#include <iostream>

```

[2]This might make you wonder why we can write the expression `r + s` in Program 33, without mentioning the namespace in which the `operator+` in question is defined. The Details of Section 4.1 explains this in the paragraph on argument-dependent name lookup.

```
namespace ifm {

  // the new type rational
  struct rational {
    int n;
    int d; // INV: d != 0
  };

  // POST: return value is the sum of a and b
  rational operator+ (rational a, rational b);

  // POST: a has been written to o
  std::ostream& operator<< (std::ostream& o, rational a);

  // POST: a has been read from i
  // PRE:  i starts with a rational number of the form "n/d"
  std::istream& operator>> (std::istream& i, rational& a);

}
```

Program 34: *progs/rational.h*

```
// Program: rational.C
// Define a type rational and operations on it

// the new type rational
struct rational {
  int n;
  int d; // INV: d != 0
};

// POST: return value is the sum of a and b
rational operator+ (rational a, rational b)
{
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}

// POST: b has been added to a; return value is the new value of a
rational& operator+= (rational& a, rational b) {
  a.n = a.n * b.d + a.d * b.n;
  a.d *= b.d;
  return a;
```

```
}

// POST: a has been written to o
std::ostream& operator<< (std::ostream& o, rational a)
{
  return o << a.n << "/" << a.d;
}

// POST: a has been read from i
// PRE:  i starts with a rational number of the form "n/d"
std::istream& operator>> (std::istream& i, rational& a)
{
  char c; // separating character, e.g. '/'
  return i >> a.n >> c >> a.d;
}
```

Program 35: *progs/rational.C*

Here is an example run of the program.

```
Rational number r:
1/2
Rational number s:
1/3
Sum is 5/6.
```

4.2.5 Const-types

Let us come back to the addition operator for rational numbers from Program 35. Although this operator does *not* intend to change the values of its call parameters, the efficiency fanatic in you might suggest to speed up this operator by using reference types anyway:

```
// POST: return value is the sum of a and b
rational operator+ (rational& a, rational& b)
{
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}
```

Indeed, this version is correct and potentially faster than the previous one, since the initialization of a formal parameter is done by copying just *one* address, rather than *two* `int` values as in the member-wise copy that takes place under the call-by-value semantics.

Even if the saving is small in this example, you can imagine that member-wise copy can be pretty expensive in structs that are more elaborate than `rational`; in contrast, call by reference is fast for *all* types, even the most complicated ones.

Unfortunately, the call parameters must be lvalues under call by reference, so we can't write the expression `a + b + c`, for example, even if `a, b, c` are variables of type `rational` (why?). Still, the faster version might work in our application; it does so in Program 33, for example, since in this program, we call `operator+` with lvalue operands only.

One less obvious (and much more dangerous) problem remains, though: in passing the parameters as references, we *allow* the operator to change the values of its call parameters in the first place, even if that happens unintentionally. In functions that are larger than the above `operator+`, it can easily happen that we modify some of the call parameters simply by mistake.

Not making such mistakes is the prime responsibility of the programmer, of course, but the clever programmer calls the programming language for help whenever possible. In this spirit, the above "efficiency fix" for `operator+` is a bad move, since it introduces a new possible source of errors.

If this sounds too abstract for you, here is an example where it is simply wrong to move to call-by-reference semantics; the compiler has no chance to detect this error since it is purely semantical. Consider the unary subtraction operator for the type `rational` from Section 4.1.5.

```
// POST: return value is -a
rational operator- (rational a)
{
  a.n = -a.n;
  return a;
}
```

Changing this to

```
rational operator- (rational& a)
{
  a.n = -a.n;
  return a;
}
```

has a drastic (and undesired) consequence: the expression `-a` will still have the same value as before, but it will have the additional effect of changing the value of `a`. We have "accidentally" created a completely different operator.

As many other high-level programming languages, C++ offers a mechanism that—if properly used—allows the compiler to detect undesired changes of values as in the previous example. The idea is to *promise* that a certain value will not be changed, and then let the compiler check whether we keep our promise. In the call-by-reference version of the unary subtraction operator, the (false) promise can be given as follows, using the keyword `const`.

```
rational operator- (const rational& a)
{
```

```
  a.n = -a.n; // error: a was promised to be constant
  return a;
}
```

In compiling this variant of the operator, the compiler will issue an error message, pointing out the mistake. We can then fix it by either going back to call-by-value semantics, or by introducing a result variable like in `operator+` above.

From the strictly functional point of view, this promise mechanism is superfluous, and there are programming languages in use that don't have it (C used to be such a language, until the `const` keyword was added in 1999, motivated by its success in C++). Also, nobody forces us to make use of the promise mechanism.[3] But the whole point of high-level programming languages is to make the programmer's life easier; the compiler is our friend and can help us to avoid many time-consuming errors. The `const` mechanism is like a check digit: by providing additional redundant data (the `const` keyword), we make sure that inconsistencies in the whole data set (the program) are automatically detected.

Definition. If T is any type, then

`const` T

is the *const-qualified* type (const-type for short) of T, and T itself is the *underlying type*. The const-qualified version of T has exactly the same value range and functionality as T. The only difference is that an expression of const-type is not allowed to change its value (in other words, it is constant); this is our promise, and the compiler checks whether we keep that promise.

If we write for example

```
const int n = 5;
n = 6;
```

the compiler will issue an error message concerning the assignment `n = 6`, since `n` has the const-type `const int`.

Values of const-type must always be initialized. Writing

```
const int n; // error: uninitialized constant
```

is illegal (and makes no sense, since we can never assign a value to `n` later).

4.2.6 What exactly is constant?

Let us consider some lvalue of type `const` T. If the underlying type T is not a reference type, then the lvalue is associated with a constant *object*.[4] For example, the declaration

[3]Indeed, "Real programmers" (as described in the classic article *Real programmers don't use PASCAL*) would leave it to the "Quiche eaters" to use such a mechanism.

[4]An rvalue of the type (or of any type, for that matter), has constant value anyway.

`const int n = 5` promises that the value of the object behind the variable `n` will not be modified. We may (accidentally) try to cheat around this promise by using another name for the object, but the compiler will catch us:

```
const int n = 5;
int i& = n; // error: const-qualification is discarded
i = 6;
```

We cannot use an expression of type `const` T to initialize (or assign to) an expression of type $T\&$, since that would create a modifiable alias for an object that was promised to be constant.

On the other hand, an lvalue (actually, any expression) of type `const` $T\&$ is the alias of an object, but that object is *not* necessarily constant itself. The const-qualification in this case is merely a promise that the object's value will not be modified *through* the alias in question. Here is an example that illustrates this point.

```
int n = 5;
const int& i = n; // i becomes a non-modifiable alias of n
int& j = n;       // j becomes a modifiable alias of n
i = 6;            // error: n is modified through const-reference
j = 6;            // ok: n receives value 6
```

Here, we do not have a constant *object*, but a constant *expression* (namely `i`). An expression of type `const` $T\&$ is also called a *const-reference*.

4.2.7 Const-references

First of all, the typename `const` $T\&$ is parenthesized as `const` ($T\&$), i.e. we get constant values of reference type. Const-references are very useful and often appear in real-life code. Let us come back to our faster version of `operator+` for rational numbers. Its "safe" version is this:

```
// POST: return value is the sum of a and b
rational operator+ (const rational& a, const rational& b) {
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}
```

The fact that this compiles confirms that we are not changing the values of the formal parameters `a` or `b` within the body of this function. But there was another problem that we apparently didn't solve yet: passing parameters by reference requires lvalues as call parameters, and this severely restricts the applicability of the operator. Fortunately, this is a non-issue: const-references (in particular, formal parameters of const-reference-type) can be initialized from rvalues as well. This means that we can write

```
const int& i = 3;
```

Behind the scenes, the compiler creates a temporary object that holds the value 3, and the address of this temporary object is used to initialize the const-reference `i`. The compiler makes sure that the temporary object does not expire before the const-reference that refers to it (see the Reference Guideline on Page 251).

The same happens when a formal function parameter of const-reference-type is initialized from an rvalue.

A parameter of type `const` *T&* is therefore the all-in-one device suitable for every purpose: *if* the call parameter is an lvalue, the initialization is very efficient (only its address needs to be copied), and otherwise, we essentially fall back to call-by-value semantics.

Despite this, there are still situations where T is preferable over `const` *T&* as a parameter type. If T is a fundamental type or a struct with small memory requirements, it does not pay off to move to `const` *T&*, since the saving in handling lvalue parameters is so small (or even nonexistent) that it won't compensate for the (slightly) more costly access to the formal function parameter in the function body. Indeed, call by reference adds one indirection: to look up the value of a formal function parameter under call-by-reference semantics, we *first* have to look up its address and then look up the actual value at that address. Under call-by-value semantics, the address of the value is "hardwired" (and refers to some object on the call stack, see Section 3.2.2).

Also, it is often convenient to use the formal parameter as a local variable and modify its value (see `operator-` above); for that, its type must not be a const-type.

4.2.8 Const-types as return types.

Const-types may also appear as return types of functions, just like any other types. In that case, the `const` promises that the function call expression itself is constant.

If the return type is not a reference type, the function call expression is an rvalue and hence not modifiable anyway. In this case, the `const` keyword is legal but has no effect. Const-types therefore only make a difference if the function returns a reference.

Note that it is *not* generally valid to replace return type T by `const` *T&*; while this safely works for the formal parameter types, it can for the return type result in syntactically correct but semantically wrong code.

As an example, let's replace `rational` by `const rational&` as the return type of `operator+`:

```
const rational& operator+ (const rational& a, const rational& b) {
  rational result;
  result.n = a.n * b.d + a.d * b.n;
  result.d = a.d * b.d;
  return result;
}
```

In executing the `return` statement, the return value (in this case a const-reference) to be passed to the caller of the function is initialized with the expression `result`. Now recall that the initialization of a (const-)reference from an lvalue simply makes it an alias of the lvalue. But the lvalue in question (namely `result`) is a local variable whose memory is freed and whose address becomes invalid when the function call terminates (see Section 2.4.3 and Section 4.2.1). The consequence is that the returned reference will be the alias of an expired object, and using this reference results in undefined behavior of the program.

Errors like this are very hard to find (and we cannot reliably count on compiler warnings here), since the program *may* work as intended, for example if the memory that was associated to the expired object is not immediately reused. But on another platform, the program may behave differently or even crash.

4.2.9 When to use const?

Whenever you think about the appropriate type of a variable, a formal function parameter, or a function's return value, it is good practice to think about const-qualification at the same time. After all, you should know what you want to do with the variable, parameter, or return value (if you don't, this paragraph is even more important), so you also know whether the program needs to change its value at some point.

The basic rule to follow is this:

Const Guideline: Use const−types whenever this is possible and makes a difference. It always makes a difference in connection with reference types.

Indeed, it is more than the promise of constant value that distinguishes the type `const` $T\&$ from $T\&$: while we need lvalues to initialize and assign to objects of type $T\&$, rvalues suffice for `const` $T\&$. We have also argued that `const` $T\&$ is preferable to T in many situations, simply for efficiency reasons. You cannot ignore these facts, even if you don't care about the promise mechanism otherwise.

If T is not a reference type, then the question whether `const` T makes a difference from T has usually not such a clear answer, with one exception: in return types of functions that do not return references, the `const` keyword really makes no difference and should therefore be omitted.

In the same spirit, the `const` keyword is typically omitted for formal function parameters that are not references. In this situation, `const` is *not* redundant, though: if a formal parameter is of const-type, we promise not to use the formal parameter as a modifiable local variable. But this promise is neither necessary to prevent accidental modification of the call parameter (call by value already takes care of this), nor does it influence the outside behavior of the function in any way. In fact, if you write functions for a library (see Section 3.1.8), you better refrain from such const-type usage, as it unnecessarily restricts you: if you later decide to change the function definition, you are committed to the const-type parameter (even if this turns out to be impractical), unless you also change the header file that contains the function declaration.

Also, not all variables that could be declared `const` in a program are typically done so, simply because it makes (or appears to make) no difference in the context of the declaration. As an example, consider line 34 in Program 32: it *is* possible to declare the variable `t` as being of const-type `const rational`, but it doesn't make a difference, since this variable occurs once only afterwards, and this occurrence is just three lines below.

For concreteness, let us stipulate that a variable that is meant to have constant value should definitely get const-type if its scope spans more than 10 lines of code.

4.2.10 Goals

Dispositional. At this point, you should ...

1) understand the alias concept behind reference types and the Reference Guideline;
2) understand the difference between *call by value* and *call by reference* semantics for function parameters;
3) understand const-types and the Const Guideline.

Operational. In particular, you should be able to ...

(G1) state exact pre-and postconditions for functions involving formal parameter types or return types of reference and/or const-type;

(G2) write functions that modify (some of) their call parameters;

(G3) find syntactical and semantical errors in programs that are due to improper handling of reference types;

(G4) find syntactical and semantical errors in programs that are due to improper handling of const-types;

(G5) find the declarations in a given program whose types should be const-according to the Const Guideline.

4.2.11 Exercises

Exercise 115 *Consider the following family of functions:*

```
T foo (S i)
{
  return ++i;
}
```

with T being one of the types `int`, `int&` *and* `const int&`, *and S being one of the types* `int`, `const int`, `int&` *and* `const int&`. *(This defines 12 different functions).*

a) Find the combinations of T and S for which the resulting function definition is syntactically valid, and explain your answer.

b) *Among the combinations found in a), find the combinations of T and S for which the resulting function definition is also semantically valid, meaning that function calls always have well-defined value and effect; explain your answer.*

c) *For all combinations found in b), give precise postconditions for the corresponding function* foo.

(G1)(G3)(G4)

Exercise 116 *Write a function that swaps the values of two* int*-variables.* (G2)
For example,

```
int a = 5;
int b = 6;
// here comes your function call
std::cout << a << "\n"; // outputs 6
std::cout << b << "\n"; // outputs 5
```

Exercise 117 *We want to have a function that* normalizes *a rational number, i.e. transforms it into the unique representation in which numerator and denominator are relatively prime, and the denominator is positive. For example,*

$$\frac{21}{-14}$$

is normalized to

$$\frac{-3}{2}.$$

There are two natural versions of this function:

```
// POST: r is normalized
void normalize (rational& r);

// POST: return value is the normalization of r
rational normalize (const rational& r);
```

Implement one of them, and argue why you have chosen it over the other one.
Hint: *you may want to use the function* gcd *from Section 3.2, modified for parameters of type* int *(how does this modification look like?).* (G2)(G2)

Exercise 118 *Provide a definition of the following function.*

```
// POST: return value indicates whether the linear equation
//       a * x + b = 0 has a real solution x ; if true is
//       returned, the value s satisfies a * s + b = 0
bool solve (double a, double b, double& s);
```

Test your function in a program for at least the pairs (a, b) *from the set*

$$\{(2, 1), (0, 2), (0, 0), (3, -4)\}.$$

(G2)

Exercise 119 *Reconsider the following programs and identify the declarations (of variables or formal parameters) in which you could replace a type T by its const-version* `const` *T.*

a) *Program 1 (Page 22)*

b) *Program 7 (Page 78)*

c) *Program 28 (Page 189)*

d) *Program 29 (Page 218)*

e) *Program 32 (Page 233)*

(G5)

Exercise 120 *Find all mistakes (if any) in the following programs, and explain why these are mistakes. All programs share the following two function definitions and only differ in their* `main` *functions.*

```
int foo (int& i) {
  return i += 2;
}

const int& bar (int &i) {
  return i += 2;
}
```

a)
```
int main()
{
  const int i = 5;
  int& j = foo (i);
}
```

b)
```
int main()
{
  int i = 5;
  const int& j = foo (i);
}
```

c)
```
int main()
{
  int i = 5;
  const int& j = bar (foo (i));
}
```

d)
```
int main()
{
  int i = 5;
  const int& j = foo( bar (i));
}
```

e)
```
int main()
{
  int i = 5;
  const int j = bar (++i);
}
```

4.2.12 Challenges

Exercise 121 *The C++ standard library also contains a type for computing with* complex numbers. *A complex number where both the real and the imaginary part are* doubles *has type* std::complex<double> *(you need to* #include <complex> *in order to get this type. In order to get a a complex number with real part* r *and imaginary part* i*, you can use the expression*

```
std::complex<double>(r,i); // r and i are of type double
```

*Otherwise, complex numbers work as expected. All the standard operators (arithmetic, relational) and mathematical functions (*std::sqrt*,* std::abs*,* std::pow*...) are available. The operators also work in mixed expressions where one operand is of type* std::complex<double> *and the other one of type* double*. Of course, you can also input and output complex numbers.*

Here is the actual challenge: implement the following function for solving cubic equations over the complex numbers:

```
// POST: return value is the number of distinct (complex) solutions
//       of the cubic equation ax^3 + bx^2 + cx + d = 0. If there
//       are infinitely many solutions (a=b=c=d=0), the return
//       value is -1. Otherwise, the return value is a number n
//       from {0,1,2,3}, and the solutions are written to s1,..,sn
int solve_cubic_equation (std::complex<double> a,
                          std::complex<double> b,
                          std::complex<double> c,
                          std::complex<double> d,
```

```
                     std::complex<double>& s1,
                     std::complex<double>& s2,
                     std::complex<double>& s3);
```

Write a program that tests your function. For example, you may substitute the solutions returned by the above function into $ax^3 + bx^2 + cx + d = 0$ *and check whether the expression indeed evaluates to (approximately) zero.*

Hint: *You find the necessary theory under the keyword* Cardano's formula.

4.3 Classes

Let me tell you this in closing
I know we might seem imposing
But trust me if we ever show in your section
Believe me its for your own protection

Will Smith, Men in Black (1997)

This section introduces the concept of classes as an extension of the struct concept from Section 4.1. You will learn about data encapsulation as a distinguishing feature of classes. This feature makes type implementations more safe and flexible. You will first learn classes feature by feature for rational numbers, and then see two complete classes in connection with random number generation.

4.3.1 Encapsulation

In the previous two sections, we have defined a new struct type `rational` whose value range models the mathematical type $\mathbb{Q}$ (the set of rational numbers), and we have shown how it can be equipped with some useful functionality (arithmetic and relational operators, input and output).

To motivate the transition from structs to classes in this section (and in particular the aspect of encapsulation), let us start off with a thought experiment. Suppose you have put the struct `rational` and all the functionality that we have developed into a nice library. In Exercise **??** you have actually done this, for the very basic version of the type `rational` from Program 34 and Program 35. Now you have sold the library to a customer; let's call it RAT (*Rational Thinking Inc.*). RAT is initially happy with the functionality that the library provides, and starts working with it. But then some unpleasant issues come up.

Issue 1: Initialization is cumbersome. Some code developed at RAT needs to initialize a new variable `r` with the rational number 1/2; for this, the programmer in charge must write

```
rational r; // default-initialization of r
r.n = 1;    // assignment to data member
r.d = 2;    // assignment to data member
```

The declaration `rational r` default-initializes `r`, but the actual value of `r` must be provided through *two* assignments later. RAT tell you that they would prefer to initialize `r` from the numerator and denominator in one go, and you realize that they have a point here. Indeed, if the programmer at RAT forgets one of the assignments, `r` has undefined value (and you get to handle the bug reports). If the struct is larger (consider the example of `rational_vector_3` on page 235), the problem is amplified.

Issue 2: Invariants cannot be guaranteed. Any legal value of the type `rational` must have a nonzero denominator. You have stipulated this as an invariant in Program 34, but there is no way of enforcing this invariant. It is possible for anyone to write

```
rational r;
r.n = 1;
r.d = 0;
```

and thus violate the *integrity* of the type, the correctness of the internal representation.

You might argue that it would be quite stupid to write `r.d = 0`, and even the programmer at RAT can't be that stupid. But in RAT's application, the values of rational numbers arise from complicated computations somewhere else in the program; these computations may result in a zero denominator simply by mistake, and in allowing value 0 to be assigned to `r.d`, the mistake further propagates instead of being withdrawn from circulation (again, you get to handle the bug reports).

You think about how both issues could be addressed in the next release of the rational numbers library, and you come up with the following solution: As another piece of functionality on the type `rational`, you define a function that creates a value of type `rational` from two values of type `int`.

```
// PRE:  d != 0
// POST: return value is n/d
rational create_rational (int n, int d) {
  // somehow check here that d != 0
  rational result;
  result.n = n;
  result.d = d;
  return result;
}
```

You then advise RAT to use this function whenever they want to initialize or assign to a rational number. For example,

```
rational r = create_rational (1, 2);
```

would initialize `r` with 1/2 in one go, and at the same time make sure that the denominator is nonzero.

Such a creation function certainly makes sense for structs in general, but the two issues above don't really go away. The reason is that this safe creation can be circumvented by not using it. In fact, your advice might not have reached the programmer at RAT, and even if it did, the programmer might be too lazy (or too stubborn) to follow it. It is therefore still possible to write `rational r;` and forget about data member assignment, and it is still possible to assign 0 to `r.d`. Behind this lies in fact a much larger problem, as you discover next.

Issue 3: The internal representation cannot be changed. After having used the rational numbers library for some time, RAT approaches you with a request for a version with a larger value range, since they have observed that intermediate values sometimes overflow.

You recall the type `extended_int` from Page 235 and realize that one thing you could easily do is to change the type of numerator and denominator from `int` to `unsigned int` and store the sign of the rational number separately as a data member of type `bool`. for example like this:

```
struct rational {
  unsigned int n;    // absolute value of numerator
  unsigned int d;    // absolute value of denominator
  bool is_negative; // sign of the rational number
};
```

It is also not too hard to rewrite the library files `rational.h` and `rational.C` to reflect this change in representation.

But shortly after you have shipped the new version of your library to RAT (you have even included the safe creation function `create_rational` from above in the hope to resolve issues 1 and 2 above), you receive an angry phone call from the management of RAT: the programmer reports that although the application code still compiles with the new version of the library, *nothing* works anymore!

After taking a quick look at the application code, you suddenly realize what the problem is: the code is cluttered up with expressions of the form *expr*`.n` and *expr*`.d`, as in

```
rational r;
r.n = 1;
r.d = 2;
```

Already this particular piece of code does not work anymore: a rational number is now represented by *three* data members, but the (old) application code obviously does not initialize the (new) member of type `bool`. Now you regret not to have provided the `create_rational` function in the first place; indeed, the statement

```
rational r = create_rational (1, 2);
```

would still work, assuming that you have correctly adapted the definition of the function `create_rational` to deal with the new representation. But the problem is much more far-reaching and manifests itself in each and every occurrence of *expr*`.n` or *expr*`.d` in the application code, since the data members have changed their meaning (they might even have changed their names): in letting RAT access numerator and denominator through data members that are specific to a certain representation, you are now committed to that representation, and you can't change it without asking RAT to change its application code as well (which they will refuse, of course).

When the RAT management realizes that the new rational numbers with extended value range are useless for them, they terminate the contract with you. Disappointed as you are, you still realize that what you need to avoid such troubles in the future is *encapsulation*: a mechanism that *hides* the actual representation of the type `rational` from the customer, and at the same time offers the customer *representation-independent* ways of working with rational numbers.

In C++, encapsulation is available through the use of classes, and we start by explaining how to hide the representation of a type from the customer.[5]

4.3.2 Public and private

Here is a preliminary `class` version of `struct rational` that takes care of data hiding.

```
class rational {
private:
   int n;
   int d; // INV: d!= 0
};
```

In the same way as a `struct`, a `class` aggregates several different types into a new type, but the `class` keyword indicates that *access restriction* may occur, realized through the keywords `public` and `private`.

A data member is *public* if and only if its declaration appears somewhere after a `public:` specifier, and with no `private:` specifier in between. It is *private* otherwise. In particular, if the class definition (see Section 4.3.9 below for the precise meaning of this term) contains no `public:` specifier, all data members are private by default. In contrast, a struct is a class where all data members are public by default.[6]

If a data member is private, it cannot be accessed by customers through the member access operator. If a data member is public, there are no such restrictions. Under our above definition of `class rational`, the following will therefore not compile:

```
rational r;
r.n = 1;      // error: n is private
r.d = 2;      // error: d is private
int i = r.n; // error: n is private
```

In particular, the assignment `r.d = 0` becomes impossible (which is good), but at a (too) high price: now your customer cannot do anything with a rational number, and even you cannot implement `operator+`, say, as you used to do it in Program 35. What we are still lacking is some way of accessing the encapsulated representation. This functionality is provided by a second category of class members, namely *member functions*.

[5]In the following, the term “customer” is used in a broader sense for all programs that use the class.

[6]In fact, access specifiers may also occur in structs, but we will use a class whenever there are any access restrictions.

4.3.3 Member functions

Let us now add the missing functionality to `class rational` through member functions. It would seem natural to start with safe creation, but since there are specific member functions reserved for this purpose, let us first show two "general" member functions that grant safe access to the numerator and denominator of a rational number (we'll discuss below what `*this` and `const` mean here; and if you wonder why we can use `n` and `d` before they are declared: this is a special feature of class scope, explained in Section 4.3.9).

```
class rational {
public:
   // POST: return value is the numerator of *this
   int numerator () const
   {
     return n;
   }
   // POST: return value is the denominator of *this
   int denominator () const
   {
     return d;
   }
private:
   int n;
   int d; // INV: d!= 0
};
```

If `r` is a variable of type `rational`, for example, the customer can then write

```
int n = r.numerator();     // get numerator of r
int d = r.denominator();   // get denominator of r
```

using the member access operator as for data members. The customer can call these two functions, since they are declared `public`. Access specifiers have the same meaning for member functions as for data members: a private member function cannot be called by the customer. This kind of access to the representation is flexible, since the corresponding member functions can easily be adapted to a new representation; it is also safe, since it is not possible to change the values of the data members through the functions `numerator` and `denominator`. As a general rule of thumb, all data members of a class should be private (otherwise, you encourage the customer to access the data members, with the ugly consequences mentioned in Issue 3 above).

The implicit call argument and *this. In order to call a member function, we need an expression of the class type for which we *access* the function, and this expression (appearing before the `.`) is an *implicit call argument* whose value may or may not be modified by the function call.

Within each member function, the lvalue `*this` refers to this implicit call argument and explains the appearance of `*this` in the postconditions of the two member functions above. It does not explain why an asterisk appears in `*this`, but we will get to this later.

Const member functions. A `const` keyword after the formal argument list of a member function refers to the implicit argument `*this` and therefore promises that the member function call does not change the value (represented by the values of the data members) of `*this`. We call such a member function a *const member function*.

Member function call. The general syntax of a member function call is

expression.fname (*expression1*, ..., *expressionN*)

Here, *expression* is an expression of a class type for which a member function called *fname* is declared, *expression1*,..., *expressionN* are the call arguments, and `.` is the member access operator. In most cases, *expression* is an lvalue of the class type, typically a variable.

Access to members within member functions. Within the body of a member function `f` of a class, any member (data member of member function) of the same class can be accessed without a prefix *expr*`.`; in this case, we implicitly access it for `*this`. In our example, the expression `n` in the return statement of the member function `numerator` refers to the data member `n` of `*this`. The call `r.numerator()` therefore does what we expect: it returns the numerator of the rational number `r`.

Within member functions, we can also access members for other expressions of the same class type through the member access operator (like a customer would do it). All accesses to class members within member functions of the same class are *unrestricted*, regardless of whether the member in question is public or private. The `public:` and `private:` specifiers are only relevant for the customer, but not for member functions of the class itself.

Member functions are sometimes also referred to as *methods* of the class.

Member functions and modularization. In the spirit of Section 3.1.8, it would be useful to source out the member function definitions, in order to allow separate compilation. This works like for ordinary functions, except that in a member function definition outside of the class definition, the function name must be qualified with the class name. In the header file `rational.h` we would then write only the declarations (as usual within namespace `ifm`):

```
class rational {
public:
  // POST: return value is the numerator of *this
```

```
    int numerator () const;
    // POST: return value is the denominator of *this
    int denominator () const;
private:
    int n;
    int d; // INV: d!= 0
};
```

The matching definitions would then appear in the source code file `rational.C` (again within namespace `ifm`, and after including `rational.h`) as follows.

```
int rational::numerator () const
{
  return n;
}
int rational::denominator () const
{
  return d;
}
```

4.3.4 Constructors

A constructor is a special member function that provides safe initialization of class values. The name of a constructor coincides with the name of the class, and—this distinguishes constructors from other functions—it does not have a return type, and consequently no return value. A class usually has several constructors, and the compiler figures out which one is meant in a given context (using the rules of overloading resolution, see the Details of Section 4.1).

The syntax of a constructor definition for a class T is as follows.

T (*T1 pname1*, *T2 pname2*, ..., *TN pnameN*)
 : *name1* (*expression1*), ..., *nameM* (*expressionM*)
 block

Here, *pname1*,..., *pnameN* are the formal arguments of the constructor. In the *initializer*

: *name1* (*expression1*), ..., *nameM* (*expressionM*)

name1,..., *nameM* are data members, and *expression1*,...,*expressionM* are expressions of types whose values can be converted to the respective data member types. These values are used to initialize the data members, before *block* is executed, and in the order in which the members are declared in the class. In other words, the order in the initializer is ignored, but it is good practice to use the declaration order here as well. If a data member is not listed in the initializer, it is default-initialized. In the constructor body *block*, we can still set or change the values of some of the data members.

For the type rational, here is a constructor that initializes a rational number from two integers.

```
// PRE: d != 0
// POST: *this is initialized with numerator / denominator
rational (int numerator, int denominator)
  : n (numerator), d (denominator)
{
  // somehow check that d != 0
}
```

To use this constructor in a variable declaration, we would for example write

```
rational r (1,2); // initializes r with value 1/2
```

In general, the declaration

T x (*expression1*, ..., *expressionN*)

defines a variable *x* of type *T* and at the same time initializes it by calling the appropriate constructor with call arguments *expression1*,..., *expressionN*.

The constructor can also be called explicitly as in

```
rational r = rational (1, 2);
```

This initializes r not directly from two integers, but from an expression of type rational that is constructed by the explicit constructor call rational(1,2) (which is of type rational).

4.3.5 Default constructor

In Section 4.1.4, we have introduced the term *default-initialization* for the kind of initialization that takes place in declarations like

```
rational r;
```

For fundamental types, default-initialization leaves the value in question undefined, but for class types, the *default constructor* is automatically called to initialize the value. If present, the default constructor is the unique constructor with an empty formal argument list.

By providing a default constructor, we can thus make sure that class type values are *always* properly initialized. In case of the class rational (or any arithmetic type), default-initialization with value 0 seems to be the canonical choice, and here is the corresponding default constructor.

```
// POST: *this is initialized with 0
rational ()
  : n (0), d (1)
{}
```

In fact, we *must* provide a default constructor if we want the compiler to accept the declaration `rational r`. This makes class types safer than fundamental types, since it is not possible to circumvent a constructor call in declaring a variable.

The careful reader will notice that there must be an exception to this rule: Program 32 in Section 4.1 contains the declaration statement `rational r;` although in that program, the type `rational` is a struct without any constructors. This is in fact the only exception: for a class without any constructors, the default constructor is implicitly provided by the compiler, and it simply default-initializes the data members; if a data member is of class type, this in turn calls the default constructor of the corresponding class. This exception has been made so that structs (which C++ has inherited from its precursor C) fit into the class concept of C++.

4.3.6 User-defined conversions

Constructors with one argument play a special role: they are *user-defined conversions*. For the class `rational`, the constructor

```
// POST: *this is initialized with value i
rational (int i)
  : n (i), d (1)
{}
```

is a user-defined conversion from `int` to `rational`. Under this constructor, `int` becomes a "type whose values can be converted to `rational`". This for example means that we can provide a call argument of type `int` whenever a formal function argument of type `rational` is expected; in the implicit conversion that takes place, the converting constructor is called. With user-defined conversions, we go beyond the set of *standard conversions* that are built-in (like the one from `int` to `double`), but in contrast to the (sometimes incomplete) standard conversion rules stipulated by the C++ standard, we make the rules ourselves.

There are meaningful user-defined conversions that can't be realized by constructors. For example, if we want a conversion from `rational` to `double`, we can't add a corresponding constructor to the type `double`, since `double` is not a class type. Even conversions to some class type T might not be possible in this way: if T is not "our" type (but comes from a library, say), we cannot simply add a constructor to T. In such situations, we simply tell *our* type how its values should be converted to the target type. The conversion from `rational` to `double`, for example, could be done through a member function named `operator double` like this.

```
// POST: return value is double-approximation of *this
operator double ()
```

```
{
  return double(n)/d;
}
```

In general, the member function `operator` S has implicit return type S and induces a user-defined conversion to the type S that is automatically invoked whenever this is necessary.

4.3.7 Member operators

All functionality of rational numbers that we have previously provided through "global" functions (`operator+`, `operator+=`,...) must now be reconsidered, since directly accessing the data members is no longer possible. Instead, we will use the member functions `numerator` and `denominator` for non-modifying access to the representation, and a constructor for returning a result. Addition for example then works like this (and becomes a bit lengthy):

```
// POST: return value is the sum of a and b
rational operator+ (rational a, rational b)
{
  int rn = a.numerator() * b.denominator() +
           a.denominator() * b.numerator();
  int rd = a.denominator() * b.denominator();
  return rational (rn, rd);
}
```

But under access restrictions, there are some things that we cannot do properly through global functions. As an example, consider `operator+=`. This operator needs to change the value of a rational number, but there is no specific member function that allows us to do this. We can only simulate the change through the addition and an assignment, like this.

```
// POST: b has been added to a; return value is the new value of a
rational& operator+= (rational& a, rational b)
{
  return a = a + b;
}
```

This works but is inefficient (consider larger structs), since we first construct an intermediate result `a + b` which is subsequently copied back into `a`. In fact, `operator+=` was designed to avoid exactly this detour that we need to take now.

A better way to go is to realize `operator+` as a public member function (a *member operator*), having only one formal argument (for `b`), and `*this` taking the role of `a`. This looks as follows.

```
// POST: b has been added to *this; return value is
//       the new value of *this
```

```
rational& operator+= (rational b)
{
  n = n * b.d + d * b.n;
  d *= b.d;
  return *this;
}
```

Within this member function, there is no problem in accessing the data members directly, since the access restrictions do not apply to member functions. This version of operator+ is as efficient as the one previously used for `struct rational`, and it can in turn even serve as a basis for a more succinct implementation of operator+:

```
// POST: return value is the sum of a and b
rational operator+ (rational a, rational b)
{
  return a += b;
}
```

Prefer nonmember operators over member operators. You might argue that even operator+ should become a member function of `class rational`, and indeed, this would probably allow a slightly more efficient implementation. There is one important reason to keep this operator global, though, and this has to do with user-defined conversions.

Having the conversion from `int` to `rational` that we get through the constructor

```
// POST: *this is initialized with value i
rational (int i);
```

we can for example write expressions like `r + 2` or `2 + r`, where `r` is of type `rational`. In compiling this, the compiler automatically inserts a converting constructor call. Now, having operator+ as a member would remove the second possibility of writing `2 + r`. Why? Let's first see what happens when `r + 2` is compiled. If operator+ is a member function, then `r + 2` "means"

```
r.operator+ (2)
```

In compiling this, the compiler inserts the conversion from the call argument type `int` to the formal argument type `rational` of operator+, and everything works as expected. `2 + r`, however, would mean

```
2.operator+ (r)
```

which makes no sense whatsoever. If we write a binary operator as a member function, then the first call argument *must* be of the respective class type. Implicit conversions do not work here: they only adapt call arguments to formal argument types of concrete functions, but they cannot be expected to "find" the class whose operator+ has to be applied.

4.3.8 Nested types

There is a third category of class members, and these are *nested types*. To motivate these, let us come back to Issue 3 above, the one concerning the internal representation of rational numbers. If you think about consequently hiding the representation of a rational number from the customer, then you probably also want to hide the numerator and denominator type. As indicated in the example, these types might internally change, but in the member functions `numerator` and `denominator`, you still promise to return `int`-values.

A better solution would be to promise only a type with certain properties, by saying for example that the functions `numerator` and `denominator` return an integral type (Section 2.2.9). Then you can internally change from one integral type to a different one without annoying the customer. Technically, this can be done as follows.

```
class rational {
public:
   // nested type for numerator and denominator
   typedef int rat_int;
   ...
   // realize all functionality in terms of rat_int
   // instead of int, e.g.
   rational (rat_int numerator, rat_int denominator); // constructor
   rat_int numerator() const;                          // numerator
   ...
private:
   rat_int n;
   rat_int d; // INV: d!= 0
};
```

In customer code, this can be used for example like this.

```
typedef rational::rat_int rat_int;
rational r (1,2);
rat_int numerator = r.numerator();       // 1
rat_int denominator = r.denominator();   // 2
```

We already see one of the properties that the nested type `rational::rat_int` must have in order for this to work. For example, values of type `int` must be convertible to it. If you have set up everything cleanly, you can now for example replace the line

```
   typedef int rat_int;
```

by the lines

```
   typedef ifm::integer rat_int;
```

and thus immediately get exact rational numbers without any overflow issues.

Typedef declarations. A *typedef declaration* introduces a new name for an existing type into its scope. It does *not* introduce a new type. In fact, the new name can be used synonymously with the old name in all contexts. In the above code, we see this twice: within the class `rational`, the typedef declaration introduces a nested type `rat_int`, a new name for the type `int`. In the customer code, the class's nested type (that can be accessed using the scope operator, if the nested type declaration is public) receives a new (shorter) name.

In real-life C++ code, there are nested types of nested types of nested types,..., and typenames tend to get very long due to this. The typedef mechanism allows us to keep our code readable.

4.3.9 Class definitions

We now have seen the major ingredients of a class. Formally, a class definition has the form

```
class T {
  class−element ... class−element
};
```

where T is an identifier. The sequence of *class-element*'s may be empty. Each *class-element* is an *access specifier* (`public:` or `private:`), or a *member declaration*. A member declaration is a declaration statement that typically declares a member function, a data member, or a nested type. Collectively, these are called *members* of the class, and their names must be identifiers. A class definition introduces a new type, and this type is called a *class type*, as opposed to a fundamental type.

A member function definition is a declaration as well, but if the class definition does not contain the definition of a member function, this function must have a matching definition somewhere else (see Section 4.3.3). All member function definitions together form the *class implementation*.

Class Scope. Any member declaration of a class is said to have *class scope*. Its declarative region is the class definition. Class scope differs from local scope (Section 2.4.3) in one aspect. The potential scope of a member declaration is not only the part of the class definition "below" the declaration, but it spans the *whole* class definition, *and* the formal argument lists and bodies of all member function definitions. In short, a class member can be used "everywhere" in the class.

If two class definitions form disjoint declarative regions, there is no problem in using the same name for members of both classes.

4.3.10 Random numbers

We now have all the means to put together a complete and useful implementation of the type `rational` as a class in C++; but since we have already seen most of the necessary code in Section 4.1 and in this section, we leave the full class as Exercise 122 and continue here with a fresh class that has a little more entertainment in store.

Playing games on the computer would be pretty boring without some unpredictability: a chess program should not always come up with the same old moves in reaction to your same old moves, and in an action game, the enemies should not always pop up at the same time and location. In order to achieve unpredictability, the program typically uses a *random number generator*. This term is misleading, though, since the numbers are in reality generated according to some fixed rule, in such a way that they *appear* to be random. But for many purposes (including games), this is completely sufficient, and we call such numbers *pseudorandom*.

Linear congruential generators. A simple and widely-used technique of getting a sequence of pseudorandom numbers is the *linear congruential method*. Given a *multiplier* $a \in \mathbb{N}$, an *offset* $c \in \mathbb{N}$, a *modulus* $m \in \mathbb{N}$ and a *seed* $x_0 \in \mathbb{N}$, let us consider the sequence $x_1, x_2, \ldots$ of natural numbers defined by the rule

$$x_i = (a x_{i-1} + c) \bmod m, \quad i > 0.$$

A small example is the pseudorandom number generator `knuth8`, defined by the following parameters.

$$a = 137, \quad c = 187, \quad m = 2^8 = 256, \quad x_0 = 0.$$

The sequence $x_1, x_2, \ldots$ of numbers that we get from this is

187, 206, 249, 252, 151, 138, 149, 120, 243, 198, 177, 116, 207, 130, 77, 240, 43, 190, 105, 236, 7, 122, 5, 104, 99, 182, 33, 100, 63, 114, 189, 224, 155, 174, 217, 220, 119, 106, 117, 88, 211, 166, 145, 84, 175, 98, 45, 208, 11, 158, 73, 204, 231, 90, 229, 72, 67, 150, 1, 68, 31, 82, 157, 192, 123, 142, 185, 188, 87, 74, 85, 56, 179, 134, 113, 52, 143, 66, 13, 176, 235, 126, 41, 172, 199, 58, 197, 40, 35, 118, 225, 36, 255, 50, 125, 160, 91, 110, 153, 156, 55, 42, 53, 24, 147, 102, 81, 20, 111, 34, 237, 144, 203, 94, 9, 140, 167, 26, 165, 8, 3, 86, 193, 4, 223, 18, 93, 128, 59, 78, 121, 124, 23, 10, 21, 248, 115, 70, 49, 244, 79, 2, 205, 112, 171, 62, 233, 108, 135, 250, 133, 232, 227, 54, 161, 228, 191, 242, 61, 96, 27, 46, 89, 92, 247, 234, 245, 216, 83, 38, 17, 212, 47, 226, 173, 80, 139, 30, 201, 76, 103, 218, 101, 200, 195, 22, 129, 196, 159, 210, 29, 64, 251, 14, 57, 60, 215, 202, 213, 184, 51, 6, 241, 180, 15, 194, 141, 48, 107, 254, 169, 44, 71, 186, 69, 168, 163, 246, 97, 164, 127, 178, 253, 32, 219, 238, 25, 28, 183, 170, 181, 152, 19, 230, 209, 148, 239, 162, 109, 16, 75, 222, 137, 12, 39, 154, 37, 136, 131, 214, 65, 132, 95, 146, 221, 0, 187, ...

From here on, the sequence repeats itself (in general, the period can never be longer than m). But until this point, it appears to be pretty random (although a closer look reveals that it is not random at all; do you discover a striking sign of nonrandomness?).

In order to make the magnitude of the random numbers independent from the modulus, it is common practice to *normalize* the numbers so that they are real numbers in the interval $[0, 1)$.

Program 36 below contains the definition of a class `random` in namespace `ifm` for generating normalized pseudorandom numbers according to the linear congruential method. There is a constructor that allows the customer to provide the parameters a, c, m, x_0, and a member function `operator()` to get the respective next element in the sequence of the x_i.

```
// Prog: random.h
// define a class for pseudorandom numbers.

namespace ifm {
  // class random: definition
  class random {
  public:
    // POST: *this is initialized with the linear congruential
    //       random number generator
    //           x_i = ( a * x_{i-1} + c) mod m
    //       with seed x0.
    random(unsigned int a, unsigned int c,
           unsigned int m, unsigned int x0);

    // POST: return value is the next pseudorandom number
    //       in the sequence of the x_i, divided by m
    double operator()();

  private:
    const unsigned int a_; // multiplier
    const unsigned int c_; // offset
    const unsigned int m_; // modulus
    unsigned int xi_;      // current sequence element
  };
} // end namespace ifm
```

Program 36: *progs/random.h*

The function `operator()` has no arguments in our case (that's why its declaration is `operator()()`, which admittedly looks a bit funny), and it overloads the *function call operator*, see Table 9 in the Appendix. In general, if *expr* is an expression of some class type which has the member function operator()(*T1 name1*,..., *TN nameN*)

then *expr* can be used like a function: the expression

expr (*expr1*,..., *exprN*)

is equivalent to a call of the member function `operator()` with arguments *expr1*,...,*exprN* for the expression *expr*. We will see such calls in Program 39 and Program 40 below.

Here is the implementation of the class `random` in which we see how `operator()` updates the value of the data member `x_i` to be the respective next element in the sequence of the x_i.

```
// Prog: random.C
// implement a class for pseudorandom numbers.

#include <IFM/random.h>

namespace ifm {
  // class random: implementation
  random::random(unsigned int a, unsigned int c,
                 unsigned int m, unsigned int x0)
    : a_(a), c_(c), m_(m), xi_(x0)
  {}

  double random::operator()()
  {
    // update xi acording to formula,...
    xi_ = (a_ * xi_ + c_) % m_;
    // ...normalize it to [0,1), and return it
    return double(xi_) / m_;
  }
} // end namespace ifm
```

Program 37: *progs/random.C*

Many commonly used random number generators are obtained in exactly this way. For example, the well-known generator `drand48` returns pseudorandom numbers in $[0, 1)$ according to the parameters

$$a = 25214903917, \quad c = 11, \quad m = 2^{48},$$

and a seed chosen by the customer.[7] It is clear that we need a large modulus to obtain a useful generator, since m is an upper bound for the number of different numbers that

[7] Assuming that the value range of `unsigned int` is $\{0, \ldots, 2^{32}-1\}$, we can't realize this generator using our class `random`. Doing all the computations over the type `double` and simulating the modulo operator in a suitable way is the way to go here.

we can possibly get from the generator. This means that knuth8 from above is rather a toy generator.

The game of choosing numbers. Here is a game that you could play with your friend while waiting for a delayed train. Each of you independently writes down an integer between 1 and 6. Then the numbers are compared. If they are equal, the game is a draw. If the numbers differ by one, the player with the smaller number gets CHF 2 from the one with the larger number. If the two numbers differ by two or more, the player with the larger number gets CHF 1 from the one with the smaller number. You can repeat this until the train arrives (or until one of you runs out of cash, and hopefully it's your friend).

If you think about how to play this game, it's not obvious what to do. One thing *is* obvious, though: you should not write down the same number in every round, since then your friend quickly learns to exploit this by writing down a number that beats your number (by design of the game, this is always possible).

You should therefore add some unpredictability to your choices. You could, for example, secretly roll a dice in every round and write down the number that it shows. But Exercise 127 reveals that your friend can exploit this as well.

You must somehow finetune your random choices, but how? In order to experiment with different distributions, you decide to define and implement a class loaded_dice that rolls the dice in such a way that the probability for number i to come up is equal to a prespecified value p_i (a fair dice has $p_i = 1/6$ for all $i \in \{1, \ldots, 6\}$). Then you could let different loaded dices play against each other, and in this way discover suitable probabilities to use against your friend (who is by the way not studying computer science).

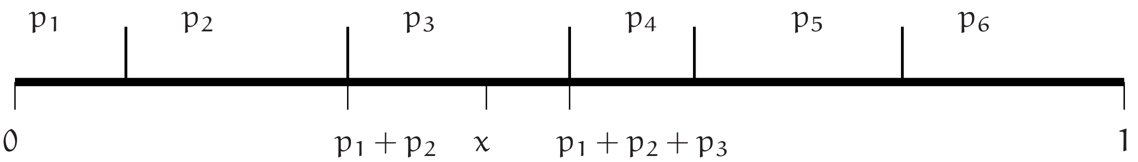
Program 38 shows a suitable class definition (that in turn relies on the class random from above, with the normalization to the interval $[0, 1)$). We will get to the class implementation (and the meaning of the data members) in Program 39 below.

```
// Prog: loaded_dice.h
// define a class for rolling a loaded dice.

#include <IFM/random.h>

namespace ifm {
  // class loaded_dice: definition
  class loaded_dice {
  public:
    // PRE: p1 + p2 + p3 + p4 + p5 <= 1
    // POST: *this is initialized to choose the number
    //       i in {1,...,6} with probability pi, according
    //       to the provided random number generator; here,
    //       p6 = 1 - p1 - p2 - p3 - p4 - p5
    loaded_dice (double p1, double p2, double p3, double p4,
```

```
                      double p5, ifm::random& generator);

    // POST: return value is the outcome of rolling a loaded
    //       dice, according to the probability distribution
    //       induced by p1,...,p6
    unsigned int operator()();

  private:
    // p_upto_i is p1 + ... + pi
    const double p_upto_1;
    const double p_upto_2;
    const double p_upto_3;
    const double p_upto_4;
    const double p_upto_5;
    // the generator (we store an alias in order to allow
    // several instances to share the same generator)
    ifm::random& g;
  };
} // end namespace ifm
```

Program 38: *progs/loaded_dice.h*

To initialize the loaded dice, we have to provide the probabilities $p_1, \ldots, p_5$ ($p_6 = 1 - \sum_{i=1}^{5} p_i$), and the random number generator that is being used to actually roll the dice. Again, we overload `operator()` to realize the functionality of rolling the dice once. How do we implement this functionality? We partition the interval $[0, 1)$ into 6 right-open intervals, where interval i has length p_i:



Then we draw a number x at random from $[0, 1)$, using our generator. If the number that we get were truly random, then it would end up in interval i with probability exactly p_i. Under the assumption that our pseudorandom numbers behave like random numbers in a suitable way, we therefore declare i as the outcome of rolling the dice if and only if x ends up in interval i. This is the case if and only if

$$p_1 + \ldots + p_{i-1} \leq x < p_1 + \ldots + p_i.$$

This explains the data members `p_upto_1`,..., `p_upto_5` (we don't need `p_upto_0` $(= 0)$ and `p_upto_6` $(= 1)$). The constructor in Program 39 simply sets these members from the data provided, and the implementation of `operator()` uses them in exactly the way that was envisioned by the previous equation.

```
// Prog: loaded_dice.C
// implement a class for rolling a loaded dice.

#include <IFM/loaded_dice.h>

namespace ifm {
  // class loaded_dice: implementation
  loaded_dice::loaded_dice
  (double p1, double p2, double p3, double p4, double p5,
   ifm::random& generator)
    : p_upto_1 (p1),
      p_upto_2 (p_upto_1 + p2),
      p_upto_3 (p_upto_2 + p3),
      p_upto_4 (p_upto_3 + p4),
      p_upto_5 (p_upto_4 + p5),
      g (generator)
  {}

  unsigned int loaded_dice::operator()()
  {
    double x = g();
    if (x <= p_upto_1) return 1;
    if (x <= p_upto_2) return 2;
    if (x <= p_upto_3) return 3;
    if (x <= p_upto_4) return 4;
    if (x <= p_upto_5) return 5;
    return 6;
  }
} // end namespace ifm
```

Program 39: *progs/loaded_dice.C*

Now you can compare two different loaded dices to find out which one is better in the game of choosing numbers. Program 40 does this, assuming that you are using a loaded dice that prefers larger numbers, and your friend uses a loaded dice that stays more in the middle. It turns out that in this setting, you win in the long run, but not by much (CHF 0.12 on average per round). Exercise 128 challenges you to find the best loaded dice that you could possibly use in this game.

```
// Prog: choosing_numbers.C
// let your loaded dice play against your friend's dice
// in the game of choosing numbers.

#include <iostream>
#include <IFM/loaded_dice.h>
```

```

// POST: return value is the payoff to you (possibly negative),
//       given the numbers of you and your friend
int your_payoff (unsigned int you, unsigned int your_friend)
{
  if (you == your_friend) return 0;       // draw
  if (you < your_friend) {
    if (you + 1 == your_friend) return 2; // you win 2
    return -1;                            // you lose 1
  } // now we have your_friend < you
  if (your_friend + 1 == you) return -2;  // you lose 2
  return 1;                               // you win 1
}

int main() {
  // the random number generator; let us use the generator
  // ANSIC instead of the toy generator knuth8; m = 2^31;
  ifm::random ansic (1103515245u, 12345u, 2147483648u, 12345u);

  // your strategy may be to prefer larger numbers and use
  // the distribution (1/21, 2/21, 3/21, 4/21, 5/21, 6/21)
  double p = 1.0/21.0;
  ifm::loaded_dice you (p, 2*p, 3*p, 4*p, 5*p, ansic);

  // your friend's strategy may be to stay more in the middle
  // and use the distribution (1/12, 2/12, 3/12, 3/12, 2/12, 1/12)
  double q = 1.0/12.0;
  ifm::loaded_dice your_friend (q, 2*q, 3*q, 3*q, 2*q, ansic);

  // now simulate 1 million rounds (the train may be very late...)
  int your_total_payoff = 0;
  for (unsigned int round = 0; round < 1000000; round++) {
    your_total_payoff += your_payoff (you(), your_friend());
  }

  // output the result:
  std::cout << "Your total payoff is "
            << your_total_payoff << "\n";

  return 0;
}
```

Program 40: *progs/choosing_numbers.C*

4.3.11 Details

Friend functions. Sometimes, we want to grant nonmember functions access to the internal representation of a class. Typical functions for which this makes sense are the in- and output operators `operator<<` and `operator>>`. Indeed, writing out or reading into the internal representation often requires some knowledge of this representation that goes beyond what other functions need.

We cannot reasonably write `operator<<` and `operator>>` as members (why not?), but we can make these functions *friends* of the class. As a friend, a function has unrestricted access to the private class members. It is clear that the *class* must declare a function to be its friend, and not the other way around, since it's the class that has to protect its privacy, and not the function. Formally, a *friend declaration* is a member declaration of the form

`friend` *function−declaration*;

This declaration makes the respective function a friend of the class and grants access to all data members, whether they are public or private. For the class `rational`, we could rewrite the `private` section as follows to declare in- and output operators to be friends of the class.

```
class rational {
private:
  friend std::ostream& operator<< (std::ostream& o, rational r);
  friend std::istream& operator>> (std::istream& i, rational& r);
  int n;
  int d; // INV: d != 0
};
```

In the definition of these operators, we can then access the numerator and denominator through `.n` and `.d` as we used to do it in Section 4.2.4. If possible, friend declarations should be avoided, since they compromise encapsulation; but sometimes, they are useful in order to save unnecessary member functions.

4.3.12 Goals

Dispositional. At this point, you should ...

1) be able to explain the purpose of a class in C++;

2) understand the new syntactical and semantical terms associated with C++ classes, in particular *access specifiers*, *member functions*, and *constructors*;

3) understand the classes `ifm::random` and `ifm::loaded_dice` in detail.

Operational. In particular, you should be able to ...

(G1) find syntactical and semantical errors in a given class definition and implementation;

(G2) describe value range and functionality of a type given by a class definition;

(G3) add functionality to a given class through member functions;

(G4) write simple classes on your own;

(G5) work with and argue about pseudorandom numbers.

4.3.13 Exercises

Exercise 122 *Provide a full implementation of rational numbers as a class type, and test it. The type should offer all arithmetic operators (including in- and decrement, and the arithmetic assignments), relational operators, as well as in- and output and user-defined conversions (from* `int` *and to* `double`*). As an invariant, it should hold that the internal representation is normalized (see also Exercise 117). For all the functionality you provide, decide whether it should be realized by member functions, or by nonmember functions. The class should also have a nested numerator and denominator type to achieve more flexibility, and there should be a conversion function from values of this type.* (G3)(G4)

Exercise 123 *Rewrite the struct* `Tribool` *that you have developed in Exercise 108 into a class, by*

a) making the data members private,

b) adding corresponding access functions,

c) adding an access function `is_bool() const` *that returns true if and only if the value is not unknown, and*

d) adding user-defined conversions from and to the type `bool`*.*

(G4)

Exercise 124

a) Find all errors in the following program. Fix them and describe the functionality of the type `Clock`*, by providing pre-and postconditions for the member functions.* (G1)(G2)

```cpp
#include <iostream>

class Clock {
  Clock(unsigned int h, unsigned int m, unsigned int s);
  void tick();
  void time(unsigned int h, unsigned int m,
            unsigned int s);
private:
  unsigned int h_;
  unsigned int m_;
  unsigned int s_;
};

Clock::Clock(unsigned int& h,
             unsigned int& m,
             unsigned int& s)
    : h_(h), m_(m), s_(s)
{}

void Clock::tick()
{
  h_ += (m_ += (s_ += 1) / 60) / 60;
  h_ %= 24; m_ %= 60; s_ %= 60;
}

void Clock::time(unsigned int& h,
                 unsigned int& m,
                 unsigned int& s)
{
  h = h_;
  m = m_;
  s = s_;
}

int main() {
  Clock c1 (23, 59, 58);
  tick();

  unsigned int h;
  unsigned int m;
  unsigned int s;
  time(h, m, s);

  std::cout << h << ":" << m << ":" << s << "\n";

  return 0;
}
```

b) Implement an output operator for the class `Clock`. (G3)

Exercise 125 *Write a program* `random_triangle.C` *to simulate the following random process graphically. Consider a fixed triangle* `t` *and choose an arbitrary vertex of* `t` *as a starting point. In each step, choose as a next point the midpoint between the current point and a (uniformly) randomly selected vertex of* `t`*.*

The simulation at each step draws the current point into a Window. Use the window object `ifm::wio` *defined in* `<IFM/window>` *for graphical output, and choose the triangle with vertices* $(0, 0)$*,* $(512, 0)$*, and* $(256, 512)$*. Use the random number generator* `ansic` *from Program 40. At begin, the program should read in a seed for* *the random number generator and the number of simulation steps to perform. For testing purposes, let the simulation run for about 100,000 steps.* (G5)

Exercise 126 *Consider the generator* `ansic` *used in Program 40. Since the modulus is* $m = 2^{31}$, *the internal computations of the generator will certainly overflow if 32 bits are used to represent* `unsigned int` *values. Despite this, the sequence of pseudorandom numbers computed by the generator is correct and coincides with its mathematical definition. Explain this!* (G5)

Exercise 127 *Find a loaded dice that beats the fair dice in the game of choosing numbers. (This is a theory exercise.)* (G5)

4.3.14 Challenges

Exercise 128 *What is the best loaded dice for playing the game of choosing numbers? Give its distribution! You could try to approximate the distribution experimentally, or somehow compute it. (***Hint***: in order to find a suitable theoretical model, search for the term "zero-sum games", or directly go to the corresponding chapter in* `http://www.inf.ethz.ch/personal/gaertner/cv/lecturenotes/ra.pdf`*. Once you have formulated the problem as a zero-sum game, you can solve it using for example the web-interface* `http://banach.lse.ac.uk/form.html` (G5)

Appendix A

C++ Operators

Description	Operator	Arity	Prec.	Assoc.
scope	::	2	18	right
subscript	[]	2	17	left
function call	()	2	17	left
construction	type()	1	17	right
member access	.	2	17	left
member access	->	2	17	left
post-increment	++	1	17	left
post-decrement	--	1	17	left
dynamic cast	dynamic_cast<>	1	17	right
static cast	static_cast<>	1	17	right
reinterpret cast	reinterpret_cast<>	1	17	right
const cast	const_cast<>	1	17	right
type identification	typeid	1	17	right
pre-increment	++	1	16	right
pre-decrement	--	1	16	right
dereference	*	1	16	right
address	&	1	16	right
bitwise complement	~	1	16	right
logical not	!	1	16	right
sign	+	1	16	right
sign	-	1	16	right
sizeof	sizeof	1	16	right
new	new	1	16	right
delete	delete	1	16	right
cast	(type)	1	16	right
member pointer	->*	2	15	left
member pointer	.*	2	15	left
	...			

	...			
multiplication	*	2	14	left
division (integer)	/	2	14	left
modulus	%	2	14	left
addition	+	2	13	left
subtraction	-	2	13	left
output/left shift	<<	2	12	left
input/right shift	>>	2	12	left
less	<	2	11	left
greater	>	2	11	left
less equal	<=	2	11	left
greater equal	>=	2	11	left
equality	==	2	10	left
inequality	!=	2	10	left
bitwise and	&	2	9	left
bitwise xor	^	2	8	left
bitwise or	|	2	7	left
logical and	&&	2	6	left
logical or	||	2	5	left
assignment	=	2	4	right
mult assignment	*=	2	4	right
div assignment	/=	2	4	right
mod assignment	%=	2	4	right
add assignment	+=	2	4	right
sub assignment	-=	2	4	right
rshift assignment	>>=	2	4	right
lshift assignment	<<=	2	4	right
and assignment	&=	2	4	right
xor assignment	^=	2	4	right
or assignment	|=	2	4	right
selection	?	3	3	right
exception	throw	1	2	right
sequencing	,	2	1	left

Table 9: *Precedences and associativities of C++ operators.*

Index