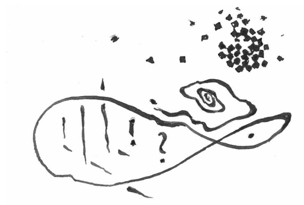

Little progress would be made in the world
if we were always afraid of possible negative consequences.

Georg Christoph Lichtenberg

Chapter 3

Infinity Is Not Equal to Infinity, or Why Infinity Is Infinitely Important in Computer Science

3.1 Why Do We Need Infinity?

The known universe is finite, and most physical theories consider the world to be finite. Everything we see and each object we touch is finite. Whatever we do in reality, we get in contact with finite things only.

> *Why then deal with infinity? Is infinity not something artificial, simply a toy of mathematics?*

In spite of possible doubts that may appear when we meet the concept of infinity for the first time, we claim that infinity is an unavoidable instrument for the successful investigation of our finite world. We touch infinity for the first time in elementary school, where we meet the set

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

of natural numbers (nonnegative integers). The concept of this set can be formulated as follows:

> For each natural number i, there is a larger natural number $i + 1$.

In other words, there does not exist any number that is larger than all other numbers (i.e., there exists no largest number), because for each number x there are numbers larger than x. What is the consequence of this concept? We are unable to write down the list of all natural numbers. It does not matter how many of them we have written already, there are still many missing. Hence, our writing is a never-ending story, and because of this we speak about **potential infinity** or about an **unbounded** number of natural numbers. We have a similar situation with the idea (the notion) of a line in geometry. Any line is potentially infinite, and so its length is unbounded (infinitely large). One can walk along a line for an arbitrarily long time and one never reaches the end; it does not matter which point (position) of the line you have reached, you can always continue to walk further in the same direction.

The main trouble with understanding the concept of infinity is that we are not capable of imagining any infinite object at once. We simply cannot see **actual infinity**. We realize that we have infinitely (unboundedly) many natural numbers, but we are not able to see all natural numbers at once. Similarly we are unable to see a whole infinite line at once. We are only able to see a finite fraction (part) of an infinite object. The way out we use is to denote infinite objects by symbols and then to work with these symbols as finite representations of the corresponding infinite objects.

To omit infinity, one can propose exchanging unbounded sizes with a huge finite bound. For instance, one can take the number[1] of all protons in Universe as the largest number and forbid all larger numbers. For most calculations and considerations one can be successful with this strategy. But not if you try to compute the whole energy of Universe or if you want to investigate all possible relations between particles of Universe. It does not matter what huge

[1] This number consists of 79 decimal digits.

number one chooses as the largest number allowed, there appear reasonable situations, whose investigation requires us to perform calculations with numbers larger than the upper bound proposed. Moreover, for every number x, we are not only aware of the existence of a number larger than x, we are even able to write this larger number down and see it as a concrete object. Why should we forbid something we can imagine (and thus has a concrete representation in our mind) and that we may even need?

To convince the reader of the usefulness of the concept of infinity, we need to provide more arguments than presenting the natural existence of potential infinity. We claim that by means of the concept of infinity we are able to investigate the world more successfully than without, and so that infinity contributes to a better understanding of the finite world around. Infinity does more than enable us to deal with infinitely large sizes; we can also consider infinitely small sizes.

What is the smallest positive rational number, i.e., what is the smallest positive fraction larger than 0?

Consider the fraction $1/1000$. We can halve it and get the fraction $1/2000$, which is smaller than $1/1000$. Now we can halve the resulting fraction again and get $1/4000 \ldots$. It does matter which small positive fraction

$$\frac{1}{x}$$

one takes, by halving it one gets the positive fraction

$$\frac{1}{2x}.$$

This fraction $1/2x$ is smaller than $1/x$ and surely still larger than 0. We see that this procedure of creating smaller and smaller numbers does not have any end too. For each positive number, there exists a smaller positive number, etc.

David Hilbert (1862–1943), one of the most famous mathematicians, said:

> *"In some sense, the mathematical analysis is nothing else than a symphony about the topic of infinity."*

We add to this quotation that current physics as we know it would not exist without the notion of infinity. The key concepts and notions of mathematics such as derivation, limit, integral and differential equations would not exist without infinity. How can physics model the world without these notions? It is unimaginable. One would already get troubles by building fundamental notions of physics. How can one define acceleration without these mathematical concepts? Many of the notions and concepts of mathematics were created because physics had a strong need to introduce and to use them.

The conclusion is that large parts of mathematics would disappear if infinity were forbidden. Since mathematics is the formal language of science, and we often measure a degree of "maturity" of scientific disciplines with respect of using this language, the exclusion of the notion of infinity would set science back several hundred years.

We have the same situation in computer science where we have to distinguish between programs, which allow infinite computations, and algorithms, which guarantee a finite computation on each input. There are infinitely many programs and infinitely many algorithmic tasks. A typical computing problem consists of infinitely many problem instances. Infinity is everywhere in computer science, and so computer scientists cannot live without this concept.

The goal of this chapter is not only to show that the concept of infinity is a research instrument of computer science. Our effort will be strange because we do not satisfy ourselves with troubles that appear when we are dealing with potential infinity and actual infinity (which no one has ever seen). We will still continue to pose the following strange question:

> *Does there exist only one infinity or do there exist several differently large infinities?*

Dealing with this question, that seems to be stupid and too abstract at first, was and is of enormous usefulness for science. Here we follow some of the most important discoveries about infinity in order to show that there exist at least two differently sized infinities. What is the gain of this? We can use this knowledge to show that the number of algorithmic problems (computing tasks) is larger than the number of all programs. In this way we obtain the first fundamental discovery of computer science.

> *One cannot automate everything. There are tasks for which no algorithm exists and so which cannot be automatically solved by any computer or robot.*

As a result of this discovery we are able to present in the next chapter concrete problems from practice that are not algorithmically (automatically) solvable. This is a wonderful example showing how the concept of an object that does not exist in the real world can help to achieve results and discoveries that are of practical importance. Remember, using hypothetical and abstract objects in research is rather typical than exceptional. And the most important thing is whether the research goal was achieved. The success is the measure of usefulness of new concepts.

3.2 Cantor's Concept for Comparing the Sizes of Infinite Sets

Comparing finite numbers is simple. All numbers lay on the real axis in increasing order from left to right. The smaller of two numbers is always to the left of the other one (Fig. 3.1).

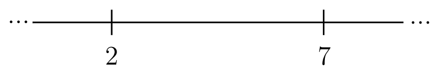

Fig. 3.1

Hence, 2 is smaller than 7 because it is to the left of 7 on the axis. But this is not a concept for comparing numbers because the numbers are a priori positioned on the axes in such a way that they increase from left to right and decrease from right to left. Though the axis is infinite in both directions, only finite numbers lay on it. It does not matter which position (which point) we consider, the number sitting there is always a concrete finite number. This is the concept of potential infinity. One can move along the axis arbitrarily far to the right or to the left, and each position reached on this trip contains a concrete finite number. There are no infinite numbers on the axis. To denote infinity in mathematics we use the symbol

$$\infty$$

called a "laying eight". Originally this symbol came from the letter aleph of the Hebraic alphabet. But if one represents infinity by just one symbol ∞, there does not exist any possibility of comparing different infinities.

What do we need to overcome this?

We need a new representation of numbers. To get it, we need the notion of a set. A set is any collection of objects (elements) that are pairwise distinct. For instance, $\{2, 3, 7\}$ is a set that contains three numbers $2, 3$, and 7. The set {John, Anna, Peter, Paula} contains four objects (elements): John, Anna, Peter, and Paula. For any set A, we use the notation

$$|A|$$

for the number of elements in A and call $|A|$ the **cardinality (size) of *A***. For instance,

$$|\{2, 3, 7\}| = 3, \text{ and } |\{\text{John, Anna, Peter, Paula}\}| = 4 .$$

Now, we take the sizes of sets as representations of numbers. In this way the cardinality of the set $\{2, 3, 7\}$ represents the integer 3, and the cardinality of the set {John, Anna, Peter, Paula} represents the number 4. Clearly, every positive integer gets a lot of different representations in this way. For instance

$$|\{1, 2\}| \text{ , } |\{7, 11\}| \text{ , } |\{\text{Petra, Paula}\}| \text{ , } |\{\square, \bigcirc\}|$$

are all representations of the integer 2. Is this not fussy? What is the gain of this seemingly too complicated representation of integers?

Maybe you find this representation to be awkward for the comparison of finite numbers.[2] But by using this way of representing numbers we gain the ability to compare infinite sizes. The cardinality

$$|\mathbb{N}|$$

for $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the infinite number that corresponds to the number of all natural numbers. If $\mathbb{Q}^+$ denotes the set of all positive rational numbers, then the number

$$|\mathbb{Q}^+|$$

represents the infinite number that corresponds to the number of all positive rational numbers (fractions). And

$$|\mathbb{R}|$$

is the infinite number that corresponds to the number of all real numbers, assuming $\mathbb{R}$ denotes the set of all real numbers. Now we see the gain. We are allowed to ask

"Is $|\mathbb{N}|$ smaller than $|\mathbb{R}|$?"

or

"Is $|\mathbb{Q}^+|$ smaller than $|\mathbb{R}|$?"

As a result of representing numbers this way we are now able to pose the question whether an infinity is larger than another infinity.

We have reduced our problem of comparing (infinite) numbers to comparing sizes of (infinite) sets. But now the following question arises:

[2] With high probability, this is the original representation of natural numbers used by Stone Age men. Small children use first the representation of numbers by sets in order to later develop an abstract concept of a "number".

How to compare the sizes of two sets?

If the sets are finite, then the comparison is simple. One simply counts the number of elements in both sets and compares the corresponding cardinalities. For sure, we cannot do this for infinite sets. If one tried to count the elements of infinite sets, then the counting would never end, and so the proper comparison would never be performed. Hence, we need a general method for comparing sizes of sets that would work for finite as well as infinite sets and that one could judge as reasonable and trustworthy. This means that we are again on the deepest axiomatic level of science. Our fundamental task is to create the notion of infinity and the definition of **"smaller than or equal to"** for the comparison of the cardinalities of two sets.

Now we let a shepherd help us. This is no shame because mathematicians did the same.

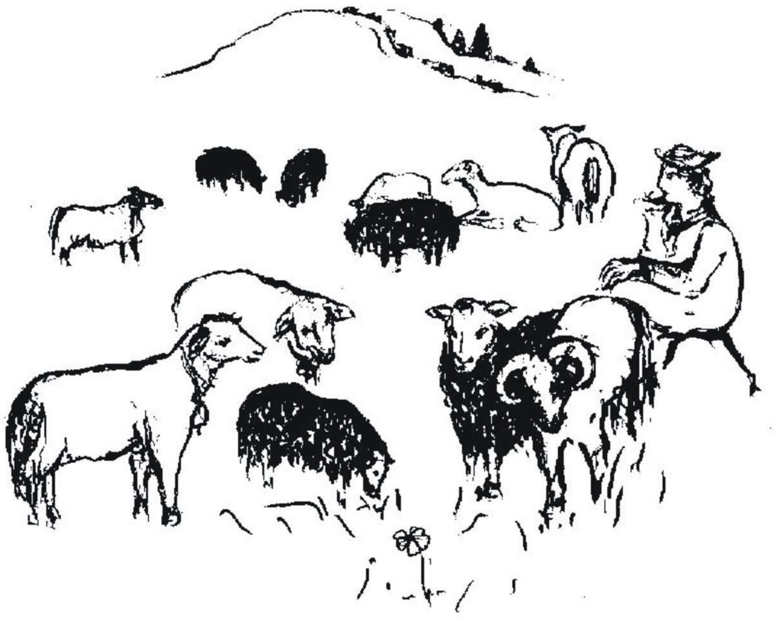

Fig. 3.2

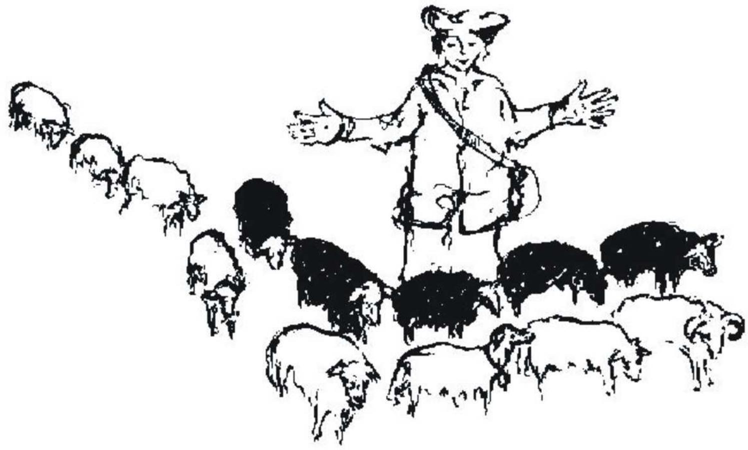

Fig. 3.3

A shepherd has a large flock of sheep with many black and white sheep. He never went to school and though his wisdom (that does not allow him to leave the mountains), he can count only to five. He wants to find out whether he has more black sheep than white ones or vice versa (Fig. 3.2).

How can he do it without counting? In the following simple and genius way. He simply takes one black sheep and one white sheep and creates one pair

(white sheep, black sheep),

and sends them away from the flock. Then he creates another white-black pair and sends it away too (Fig. 3.3). He continues in this way until he has sheep of one color only or there are no remaining sheep at all (i.e., until there is no way to build a white-black pair of sheep). Now he can reach the following conclusion :

(i) If no sheep remained, he has as many white sheep as black ones.

(ii) If one or more white sheep remained in the flock, then he has more white sheep than black ones (Fig. 3.3).

(iii) If one or more black sheep remained in the flock, then he has more black sheep than white ones.

Pairing the sheep and conclusion (i) is used by mathematicians as the base for comparing the sizes of sets.

Definition 3.1. Let A and B be two sets. A **matching** of A and B is a set of pairs (a, b) that satisfies the following rules:

(i) Element a belongs to A ($a \in A$), and element b belongs to B ($b \in B$).

(ii) Each element of A is the first element of exactly one pair (i.e., no element of A is involved in two or more pairs and no element of A remains unmatched).

(iii) Each element of B is the second element of exactly one pair.

For each pair (a, b), we say that ***a* and *b* are married**. We say that ***A* and *B* have the same size** or that **the size of *A* equals to the size of *B*** and write

$$|\boldsymbol{A}| = |\boldsymbol{B}|$$

if there exists a matching of A and B. We say that **the size of *A* is not equal to the size of *B*** and write

$$|\boldsymbol{A}| \neq |\boldsymbol{B}|$$

if there does not exist any matching of A and B.

Consider the two sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$ depicted in Fig. 3.4. Figure 3.4 depicts the matching

$$(2, 2), (3, 5), (4, 7), (5, 11) \ .$$

Each element of A is involved in exactly one pair of the matching as the first element. For instance, the element 4 of A is involved as the first element of the third pair $(4, 7)$. Each element of B is involved in exactly one pair as the second element. For instance, the element 5 of B is involved in the second pair. In other words, each element of A is married to exactly one element of B, each

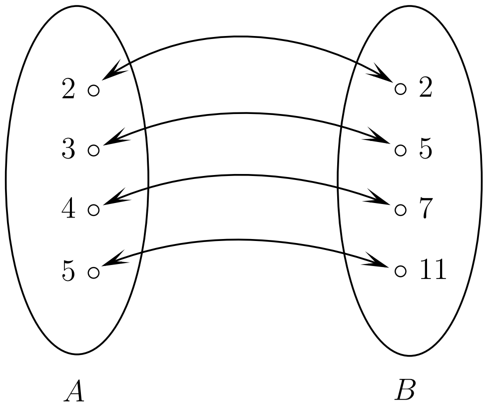

Fig. 3.4

element of B is married to exactly one element of A, and so no element of A or B remains single. Therefore, we can conclude

$$|\{2, 3, 4, 5\}| = |\{2, 5, 7, 11\}| \ .$$

You can also find other matchings of A and B. For instance,

$$(2, 11), (3, 7), (4, 5), (5, 2)$$

is also a matching of A and B.

Exercise 3.1 (a) Give two other matchings of the sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$.
(b) Why is $(2, 2), (4, 5), (5, 11), (2, 7)$ not a matching of A and B?

Following this concept of comparing the sizes of two sets, a set A of girls and a set B of boys are equally sized, if all the women and men from A and B can get married in such a way that no single remains.[3]

A matching of the sets $C = \{1, 2, 3\}$ and $D = \{2, 4, 6, 8\}$ cannot exist because every attempt to match the elements of D and

[3]Building same-sex pairs is not allowed here.

C ends in the situation where one element of D remains single. Therefore, $|D| \neq |C|$ holds. An unsuccessful attempt to match C and D is depicted in Fig. 3.5.

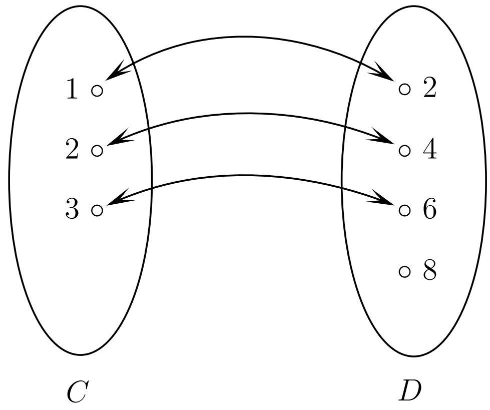

Fig. 3.5

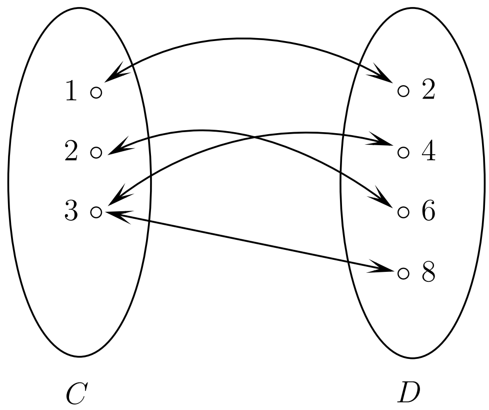

Fig. 3.6

Figure 3.6 shows another attempt to match C and D. Here the result is not a matching of C and D because element 3 of C is married to two elements 4 and 8 of D.

But we do not need the concept of matching in order to compare the sizes of finite sets. We were also able to do it without this concept. In the previous description, we only checked that our matching concept works in the finite world.[4] In what follows we try to apply this concept to infinite sets. Consider the two sets

$$\mathbb{N}_{even} = \{0, 2, 4, 6, 8, \ldots\}$$

of all even natural numbers and

$$\mathbb{N}_{odd} = \{1, 3, 5, 7, 9, \ldots\}$$

of all odd natural numbers. At the first glance, these sets look to be of the same size, and we try to verify it by the means of our concept. We match each even number $2i$ to the odd number $2i+1$.

Following Fig. 3.7, we see that we get an infinite sequence of pairs

$$(0, 1), (2, 3), (4, 5), (6, 7), \ldots, (2i, 2i + 1), \ldots$$

in this way. This sequence of pairs is a correct matching of A and B. No element from $\mathbb{N}_{even}$ or of $\mathbb{N}_{odd}$ is involved in two or more pairs (is married to more than one element). On the other hand no element remains single (unmarried). For each even number $2k$ from $\mathbb{N}_{even}$, we have the pair $(2k, 2k + 1)$. For each odd number $2m+1$ from $\mathbb{N}_{odd}$, we have the pair $(2m, 2m+1)$. Hence, we verified that the equality $|\mathbb{N}_{even}| = |\mathbb{N}_{odd}|$ holds.

Exercise 3.2 Prove that $|\mathbb{Z}^+| = |\mathbb{Z}^-|$, where $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ and $\mathbb{Z}^- = \{-1, -2, -3, -4, \ldots\}$. Draw a figure depicting your matching as we did for $\mathbb{N}_{even}$ and $\mathbb{N}_{odd}$ in Fig. 3.7.

Up to this point everything looks tidy, understandable, and acceptable. Now, we present something which may be difficult to

[4] If the concept did not work in the finite world, then we would have to reject it.

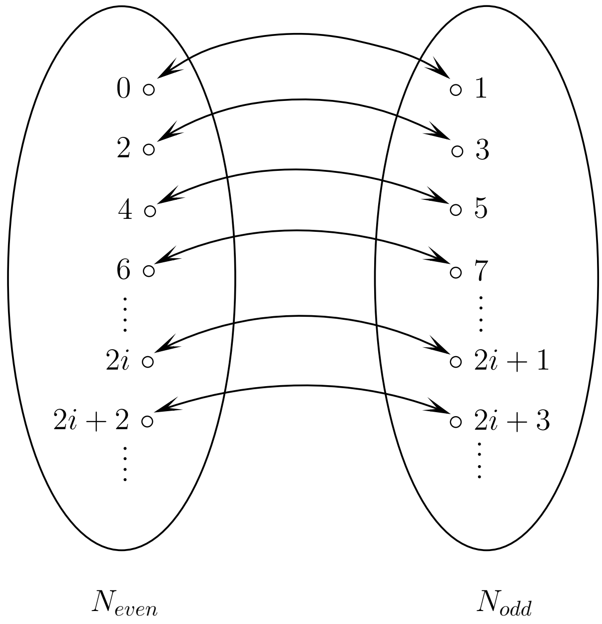

Fig. 3.7

come to terms with, at least at the first attempt. Consider the sets

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \text{ and } \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \ .$$

All elements of $\mathbb{Z}^+$ are in $\mathbb{N}$, and so

$$\mathbb{Z}^+ \subseteq \mathbb{N} \ ,$$

i.e., $\mathbb{Z}^+$ is a **subset** of $\mathbb{N}$. Moreover, the element 0 belongs to $\mathbb{N}$ ($0 \in \mathbb{N}$), but not to $\mathbb{Z}^+$ ($0 \notin \mathbb{Z}^+$). We therefore say that $\mathbb{Z}^+$ is a **proper subset** of $\mathbb{N}$ and write $\mathbb{Z}^+ \subset \mathbb{N}$. The notion "$A$ is a proper subset of B" means that A is a part of B but not the whole of B. We can see this situation transparently for the case

$$\mathbb{Z}^+ \subset \mathbb{N}$$

in Fig. 3.8. We see that $\mathbb{Z}^+$ is completely contained in $\mathbb{N}$ but $\mathbb{Z}^+$ does not cover the whole $\mathbb{N}$ because $0 \in \mathbb{N}$ and $0 \notin \mathbb{Z}^+$.

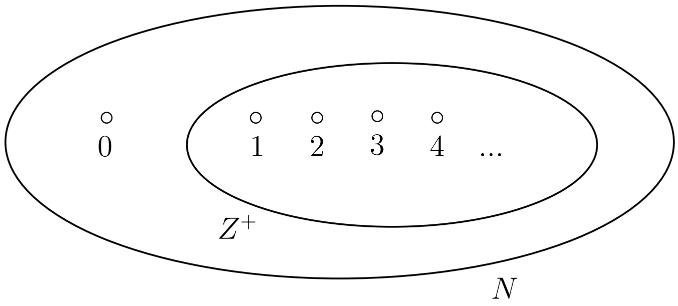

Fig. 3.8

However, we claim that

$$|\mathbb{N}| = \left|\mathbb{Z}^+\right|$$

is true, i.e., that the sizes of $|\mathbb{N}|$ and $|\mathbb{Z}^+|$ are equal. We justify this claim by building the following matching

$$(0,1),(1,2),(2,3),\ldots,(i,i+1),\ldots\ ,$$

depicted in Fig. 3.9.

We clearly see that all elements of $\mathbb{N}$ and $\mathbb{Z}^+$ are correctly married. No element remains single. The conclusion is that $\mathbb{N}$ is not larger than $\mathbb{Z}^+$ though $\mathbb{N}$ has one more element than $\mathbb{Z}^+$. But this fact may not be too surprising or even worrying. It only says that

$$\infty + 1 = \infty\ ,$$

and so that increasing infinity by 1 does not lead to a larger infinity. This does not look surprising. What is 1 in comparison with infinity? It is nothing and can be neglected. This at first glance surprising combination of the facts

$$\mathbb{Z}^+ \subset \mathbb{N} \text{ (Fig. 3.8) and } \left|\mathbb{Z}^+\right| = |\mathbb{N}| \text{ (Fig. 3.9)}$$

provides the fundamentals used for creating the mathematical definition of infinity. Mathematicians took thousands of years to find

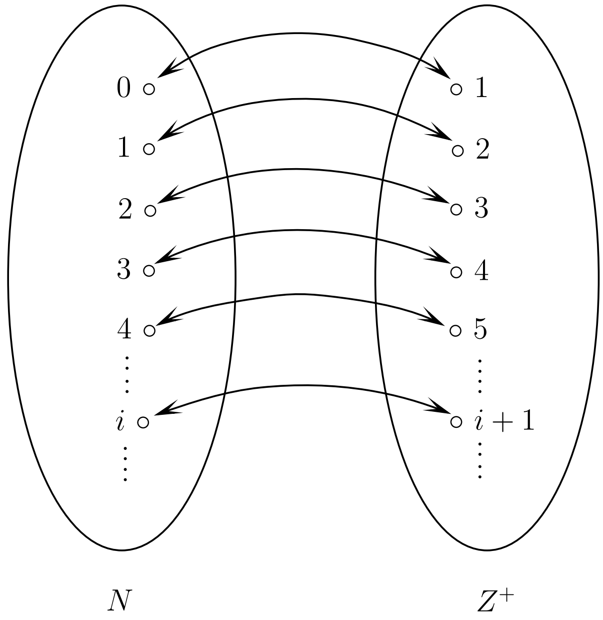

Fig. 3.9

this definition and then one generation exchange in research was needed to be able to accept it and fully imagine its meaning. It was not so easy for them to see that this definition provides what they strived for, namely a formal criterion for distinguishing between finite sets and infinite sets.

Definition 3.2. A set A is infinite if and only if there exists a proper subset B of A such that

$$|A| = |B| \ .$$

In other words:

> *An object is* ***infinite*** *if there is a proper part of the object that is as large as the whole object.*

Now you can say: *"Stop! This is too much for me. I cannot accept something like that. How can a part be of the same size as the whole? Something like this does not exist."*

It is excellent that you have this opinion. Especially because of this, this definition is good. In the real world in which everything is finite, no part can be as large as the whole. This is exactly what we can agree on. No finite (real) object can have this strange property. And in this way, Definition 3.2 says correctly that all such objects are finite (i.e., not infinite). But in the artificial world of infinity, it is not only possible to have this property, but also necessary. And so this property is exactly what we where searching for, since a thing that has this property is infinite and one that does not have this property is finite. In this way, Definition 3.2 provides a criterion for classifying objects into finite and infinite and this is exactly what one expects from such a definition.

To get a deeper understanding of this at the first glance strange property of infinite objects, we present two examples.

Example 3.1 Hotel Hilbert

Let us consider a hotel with infinitely many single rooms that is known as the Hotel Hilbert. The rooms are enumerated as follows:

$$Z(0), Z(1), Z(2), Z(3), \ldots, Z(i), \ldots \quad .$$

All rooms are occupied, i.e., there is exactly one guest in each room. Now, a new guest enters the hotel and asks the porter: "Do you have a free room for me?" "No problem", answers the porter and accommodates the new guest by the following strategy. He asks every guest in the hotel to move to the next room with the number that is 1 higher than the room number of the room used up till now. Following this request, the guest in room $Z(0)$ moves to the room $Z(1)$, the guest in $Z(1)$ moves to $Z(2)$, etc. In general, the guest in $Z(i)$ moves to the room $Z(i+1)$. In this way, the room $Z(0)$ becomes free, and so $Z(0)$ can be assigned to the newcomer (Fig. 3.10).

We observe that, after the move, every guest has her or his own room and room $Z(0)$ becomes free for the newcomer. Mathemati-

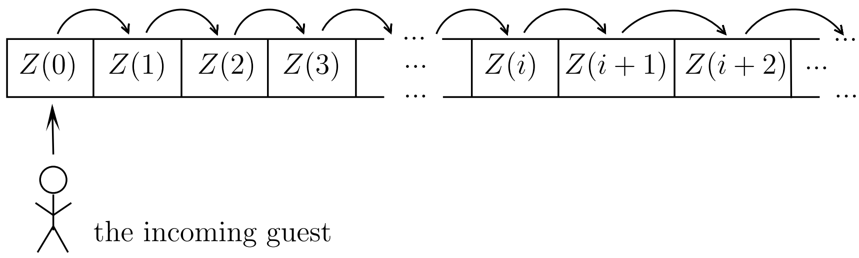

Fig. 3.10

cians argue for the truthfulness of this observation as follows. Clearly room $Z(0)$ is free after the move. The task is to show that every guest has his or her own room after the move. Let G be an arbitrary guest. This person G lives alone in a concrete room before the move. Let $Z(n)$ be the number of this room. Following the instructions of the porter, guest G moves from $Z(n)$ to $Z(n+1)$. He can do this because $Z(n+1)$ becomes free because the guest in this room moved to room $Z(n+2)$. Hence, after the moves guest G lives alone in room $Z(n+1)$. Since our argument is valid for every guest of the hotel, all guests have a single room accommodation after the move.

The solution above shows why the actual infinity was considered as a paradox[5] of mathematics for a long time. Hotel Hilbert is an actual infinity. Something like this can only be outlined by drawing a finite part of it and adding $\cdots$. But nobody can see it at once. Hence, it is not possible to observe the whole move of infinitely many guests at once. On the other hand, observing each particular guest separately, one can verify that the move works successfully.

Only when one was able to realize that infinity differs from finiteness by having proper subparts of the same size as the whole, was this paradox solved[6]. We observe that the move corresponds to

[5] a seemingly contradictory fact or an inexplicable situation
[6] and so it is not a paradox anymore

matching the elements of the set $\mathbb{N}$ (the set of guests) with the set $\mathbb{Z}^+$ (the set of rooms up to the room $Z(1)$). □

Exercise 3.3 (a) Three newcomers enter Hotel Hilbert. As usual, the hotel is completely booked. Play the role of the porter and accommodate the three new guests in such a way that no former guest has to leave the hotel and after the move, each new guest and each former guest possess their own room. If possible, arrange the accommodation using one move of each guest instead of organizing 3 moves one after each other.

(b) A newcomer enters Hotel Hilbert and asks for his favored room, $Z(7)$. How can the porter satisfy this request?

We take the next example from physics. Physicists discovered it as a remedy for depressions caused by imagining that our Earth and mankind are tiny in the comparison with the huge universe[7].

Example 3.2 Let us view our Earth and Universe as infinite sets of points of size 0 that can lie arbitrarily close each to each other. To simplify our story we view everything two dimensionally instead of working in three dimensions. The whole Universe can be viewed as a large sheet of paper, and Earth can be depicted as a small circle on the sheet (Fig. 3.11). If somebody has doubts about viewing our small Earth as an infinite set of points, remember that there are infinitely many points on the finite part of the real axis between the numbers 0 and 1. Each rational number between 0 and 1 can be viewed as a point on the line between 0 and 1. And there are infinitely many rational numbers between 0 and 1. We proved this fact already by generating infinitely many rational numbers between 0 and 1 in our unsuccessful attempt to find the smallest positive rational number.

Another justification of this fact is related to the proof of the following claim.

> For any two different rational numbers a and b, $a < b$, there are infinitely many rational numbers between a and b.

The first number between a and b we generate is the number $c_1 = \frac{a+b}{2}$, i.e., the average value of a and b. The next one is $c_2 = \frac{c_1+b}{2}$,

[7] In this way, physicists try to ease the negative consequences of their discoveries.

i.e., the average of c_1 and b. In general, the i -th generated number from $[a, b]$ is

$$c_i = \frac{c_{i-1} + b}{2},$$

i.e., the average of c_{i-1} und b. When $a = 0$ and $b = 1$, then one gets the infinite sequence

$$\frac{1}{2}\ ,\ \frac{3}{4}\ ,\ \frac{7}{8}\ ,\ \frac{15}{16}\ ,\ \ldots$$

of pairwise different rational numbers between 0 and 1.

Now let us finally switch to the fact physicists want to tell us. All points of our huge universe beyond Earth can be matched with the points of Earth. This claim has two positive (healing) interpretations:

(i) The number of points of our Earth is equal to the number of points of Universe outside Earth.

(ii) Everything what happens in Universe can be reflected on Earth and so can be imitated in our tiny world.

Hence, our task to search for a matching between the Earth points and the points outside Earth. In what follows we show how to assign an Earth point P_E to any point P_U outside Earth.

First, we connect P_U and the Earth center M by a line (Fig. 3.11). The point P_E we are searching for has to lay on this line. Next, we depict the two tangents t_1 and t_2 of the circle that goes through the point P_U (Fig. 3.11). Remember that a tangent of a circle is a line that has exactly one common point with the circle. We call the point in which t_1 touches the circle A_P and we denote by B_P the common point[8] of the circle and the line t_2 (see Fig. 3.11). Finally, we connect the points B_P and A_P by a line B_PA_P (Fig. 3.12). The point in the intersection of the lines B_PA_P and P_UM is the Earth point P_E we assign to P_U (Fig. 3.12).

[8] Mathematicians would say that the point A_P is the intersection of the circle and t_1 and that B_P is the intersection of the circle and t_2.

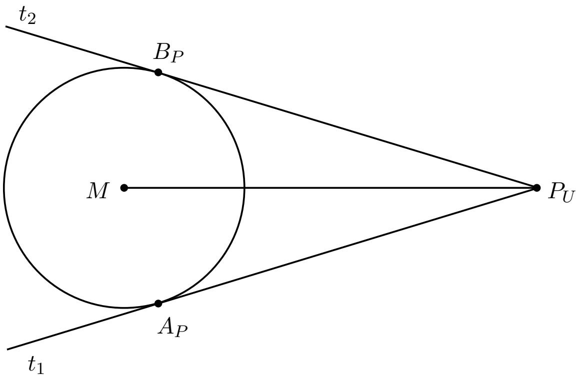

Fig. 3.11

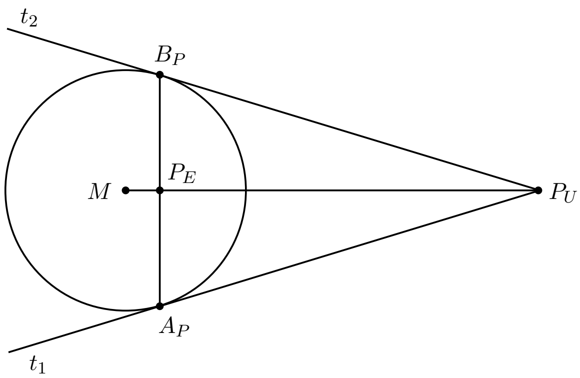

Fig. 3.12

Next, we have to show that this geometric assignment of P_E to P_U defines a matching between the Earth's points and the points outside Earth. Namely we have to show that one always assigns two distinct Earth's points P_E and P'_E to two different points P_U and P'_U outside Earth.

To verify this fact, we distinguish two possibilities with respect to the positions of P_U and P'_U according to M.

(i) The points M, P_U, and P'_U do not lie on the same line. This situation is depicted in Fig. 3.13. We know that P_E lies on the line MP_U and that P'_E lies on the line MP'_U. Since the only common point of the lines MP_U and MP'_U is M and M is different from P_E and P'_E, independently of the positions of P_E and P'_E on their lines, the points of P_E and P'_E must be different.

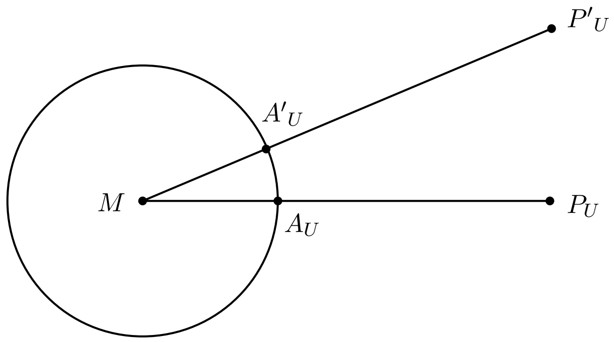

Fig. 3.13: E_U lies on MA_U and E'_U lies on MA'_U, and therefore E_U and E'_U are different points.

(ii) All three points M, P_U, and P'_U lie on the same line (Fig. 3.14). Therefore, E_U and E'_U lie on this line, too. Then, we perform our assignment construction for both points P_U and P'_U as depicted in Fig. 3.12. We immediately see in Fig. 3.14 that E_U and E'_U are different.

We showed that, independently of the fact how many times larger Universe is than Earth, the number of points in Earth is equal to the number of points in Universe outside Earth. □

Exercise 3.4 Complete Fig. 3.13 by estimating the exact positions of points P_E and P'_E.

Exercise 3.5 Consider the semicircle in Fig. 3.15 and the line AB that is the diameter of the circle. Justify geometrically as well as by calculations that the number of points of the line AB is the same as the number of points of the curve of the semicircle.

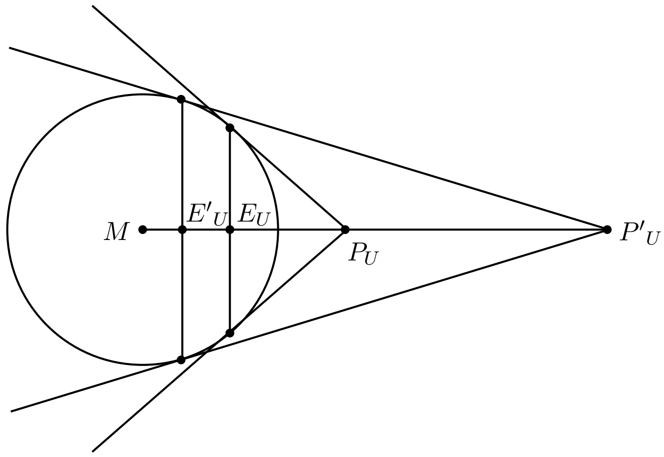

Fig. 3.14

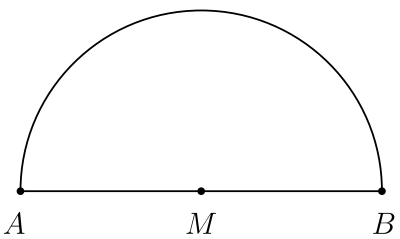

Fig. 3.15

Exercise 3.6 Consider the curve of the function F in Fig. 3.16 and the line AB. Why does this curve have as many points as the line AB?

If you still have a stomachache when trying to imagine and to accept Cantor's concept of infinity, please, do not worry. Mathematicians needed many years to develop this concept, and after discovering it, 20 years were needed to get it accepted by broad mathematical community. Take time for repeated confrontations with the definition of infinite sets. Only if one iteratively deals with this topic, can one understand why one takes over this definition

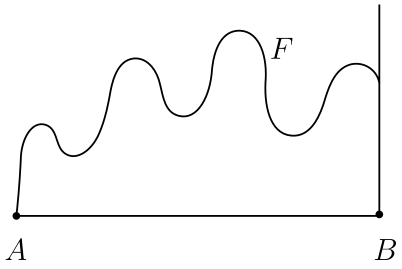

Fig. 3.16

of infinity as an axiom of mathematics and why mathematicians consider it not only trustworthy, but they even do not see any alternative to this definition.

In what follows we shortly discuss the most frequent proposal for the concept of comparing infinite sizes that some listener proposed after the first confrontation with infinity. If

$$A \subset B$$

holds (i.e., if A is a proper subset of B), then

$$|A| < |B| \ .$$

Clearly, this attempt to compare infinite sizes reflects in another way the refusal of our key idea that a part of an infinite object may be as large as the whole. This proposal for an alternative definition has two drawbacks. First, one can use it only for comparing two sets where one is a subset of the other. This definition does not provide the possibility to compare two different sets such as $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For a comparison of these two sets one has to search for another relation between them. Realizing this drawback, some listeners propose to accept the matching approach in the following way. One can find a matching between one of the sets and a subset of another one and then compare using the originally proposed subset principle. We show that one can get a nonsense in this way. Namely that

$$|\mathbb{N}| < |\mathbb{N}| \ ,$$

i.e., that $\mathbb{N}$ is smaller than $\mathbb{N}$ itself. Using the concept of matching we proved

$$|\mathbb{N}| = |\mathbb{Z}^+|. \tag{3.1}$$

Since $\mathbb{Z}^+ \subset \mathbb{N}$, using the subset principle, one gets

$$|\mathbb{Z}^+| < |\mathbb{N}|. \tag{3.2}$$

Combining Equations (3.1) and (3.2) we obtain

$$|\mathbb{N}| = |\mathbb{Z}^+| < |\mathbb{N}|,$$

and so $|\mathbb{N}| < |\mathbb{N}|$.

In this way we proved that the concept of the shepherd (of matching) and the subset principle for comparing the cardinalities of two sets contradict each other because adopting both at once leads to an obvious nonsense.

Why do we spend so much time to discuss this axiom of mathematics and why do we such a big effort to understand it? As you may already suspect, this axiom is only the beginning of our troubles. The concept of infinity is not the only surprise of this chapter. In some sense we showed $\infty = \infty + 1$ for $\infty = |\mathbb{N}|$ and also give to understand that $\infty = \infty + c$ for any finite number c. Example 3.2 and the following exercises even intimate

$$\infty = c \cdot \infty$$

for an arbitrary finite number (constant) c.

Let us consider $\mathbb{N}$ and the set

$$\mathbb{N}_{even} = \{0, 2, 4, 6, \ldots\} = \{2i \mid i \in \mathbb{N}\}$$

of all even natural numbers. At the first glance $\mathbb{N}$ contains twice as many elements as $\mathbb{N}_{even}$. In spite of this view (Fig. 3.17) one can match the elements of $\mathbb{N}$ and of $\mathbb{N}_{even}$ as follows:

$$(0,0),(1,2),(2,4),(3,6),\ldots,(i,2i),\ldots \ .$$

We see that each element of both sets is married exactly once. The immediate consequence is

$$|\mathbb{N}| = |\mathbb{N}_{even}| \ .$$

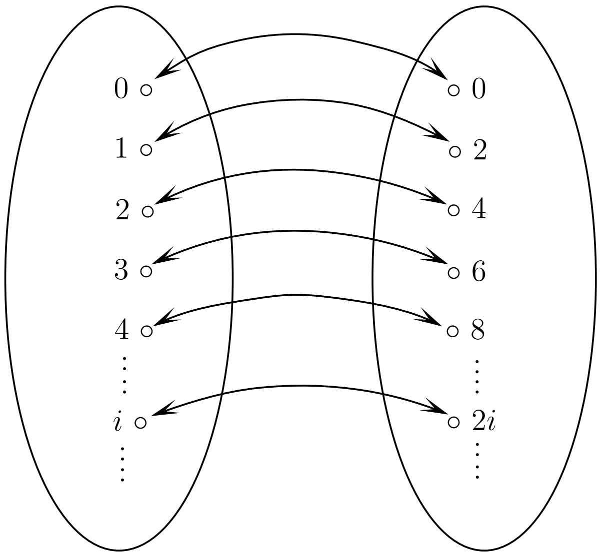

Fig. 3.17

We can explain this somewhat surprising result

$$2 \cdot \infty = \infty$$

again by a story about Hotel Hilbert.

Example 3.3 Consider once again Hotel Hilbert with infinitely many single rooms

$$Z(0), Z(1), Z(2), \ldots$$

that are all occupied by guests. Now, an infinite bus arrives. This bus has infinitely many seats

$$B(0), B(1), B(2), \dots ,$$

and all seats are occupied by passengers[9]. The bus driver asks the porter whether he can accommodate all passengers. As usual, the porter answers: "No problem", and does the following:

He asks each guest in room $Z(i)$ to move to room $Z(2i)$ as depicted in the upper part of Fig. 3.18. After the move, each former guest has her or his own room and all rooms with odd numbers $1, 3, 5, 7, \dots, 2i+1 \dots$ are empty. Now, it remains to match the free rooms with the bus passengers. The porter assigns room $Z(1)$ to the passenger sitting on seat $B(0)$, room $Z(3)$ to the passenger sitting on sit $B(1)$, etc. In general, the passenger from $B(i)$ gets room $Z(2i+1)$ as depicted in Fig. 3.18. In this way, one gets the matching

$$(B(0), Z(1)), (B(1), Z(3)), (B(2), Z(5)), \dots, (B(i), Z(2i+1)), \dots$$

between the empty rooms with odd numbers and the seats of the infinite bus.

□

Exercise 3.7 a) Hotel Hilbert is only partially occupied. All rooms $Z(0), Z(2), Z(4), \dots, Z(2i), \dots$ with even numbers are occupied and all rooms with odd numbers are free. Now, two infinite buses B_1 and B_2 arrive. The seats of the buses are numbered as follows:

$$B_1(0), B_1(1), B_1(2), B_1(3), \dots$$

$$B_2(0), B_2(1), B_2(2), B_2(3), \dots$$

How can the porter act in order to accommodate all guests? Is it possible to accommodate all newcomers without asking somebody to move to another room?

b) Hotel Hilbert is fully occupied. Now, three infinite buses are coming. The seats of each bus are enumerated by natural numbers. How can the porter accommodate everybody?

Exercise 3.8 Show by matching $\mathbb{Z}$ and $\mathbb{N}$ that

$$|\mathbb{Z}| = |\mathbb{N}|$$

holds, where $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of all integers.

[9]Each seat is occupied by exactly one passenger.

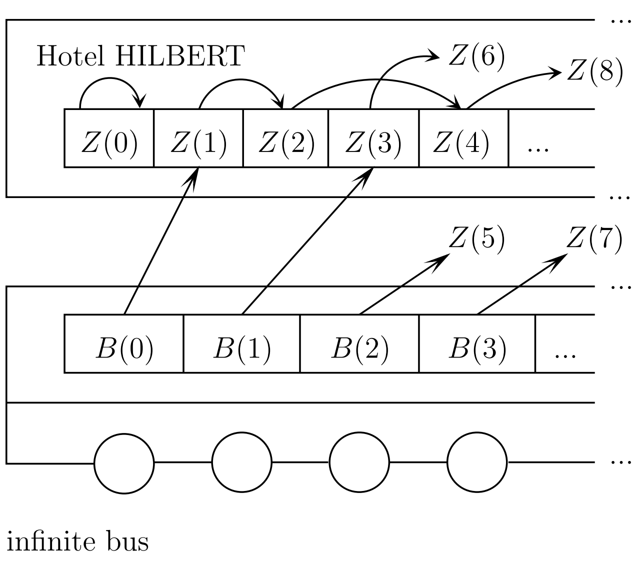

Fig. 3.18

Exercise 3.9 (challenge) Let $[a, b]$ be the set of all points (all real numbers) of the real axis between a and b.

a) Show that
$$|[0,1]| = |[1,10]| \ .$$
Try to show this by geometric means as in Example 3.2.

b) Prove
$$|[0,1]| = |[1,100]|$$
by arithmetic arguments, i.e., find a function f such that the pairs $(f(i), i)$ for $i \in [0,100]$ build a matching of $[0,1]$ and $[0,100]$.

Exercise 3.10 (challenge) Assume that Hotel Hilbert is empty, i.e., there are no guests accommodated in the hotel. Since all used accommodation strategies were based on moving former guests from a room to another, there is the risk that to stay in the hotel may become unpopular. Therefore, the porter needs an accommodation strategy that does not require any move of an already accommodated guest. This accommodation strategy has to work even if arbitrarily many finite and infinite buses arrive in arbitrarily many different moments. Can you help the porter?

We observe that proving

$$|\mathbb{N}| = |A|$$

for a set A does mean nothing else than numbering all elements of set A by natural numbers. A matching between $\mathbb{N}$ and A unambiguously assigns a natural number from $\mathbb{N}$ to each element of A. And this assigned natural number can be viewed as the order of the corresponding element of A. For instance, if $(3, \text{John})$ is a pair of the matching, then John can be viewed as the third element of set A. Vice versa, each numbering of elements of a set A directly provides a matching between $\mathbb{N}$ and A. The pair of the matching are simply

$$\text{order of } a, a$$

for each element a of A. In what follows, the notion of **numbering**[10] the elements of A enables us to present transparent arguments for claims $|\mathbb{N}| = |A|$ for some sets A, i.e., for showing that A has as many elements as $\mathbb{N}$.

The matching

$$(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3), \ldots$$

of the sets $\mathbb{N}$ and $\mathbb{Z}$ assigns the following order to the elements of $\mathbb{Z}$:

$$0, 1, -1, 2, -2, 3, -3, \ldots \ .$$

In this way 0 is the 0-th element, 1 is the first element, -1 is the second element, 2 is the third element, etc.

Exercise 3.11 Assign to $\mathbb{Z}$ an order of elements other than the one presented above by giving another matching.

Exercise 3.12 Prove that

$$|\mathbb{N}| = |\mathbb{N}_{quad}| \ ,$$

where $\mathbb{N}_{quad} = \{i^2 \mid i \in \mathbb{N}\} = \{0, 1, 4, 9, 16, 25, \ldots\}$ is the set of all squares of natural numbers. What order of the elements of $\mathbb{N}_{quad}$ do you get by the matching you proposed?

Our attempt to answer the next question increases the degree of difficulty of our considerations. What is the relation between $|\mathbb{N}|$ and $|\mathbb{Q}^+|$? Remember that

[10] In the scientific literature one usually uses the term "enumeration" of the set A.

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}^+ \right\}$$

is the set of all positive rational numbers. We have already observed that by calculating averages repeatedly one can show that there are infinitely many rational numbers between any two rational numbers a and b with $a < b$. If one partitions the real axes into infinitely many parts $[0,1], [1,2], [2,3], \ldots$ as depicted in Fig. 3.19, then the cardinality of $\mathbb{Q}^+$ looks like

$$\infty \cdot \infty = \infty^2$$

because each of these infinitely many parts (intervals) contains infinitely many rational numbers.

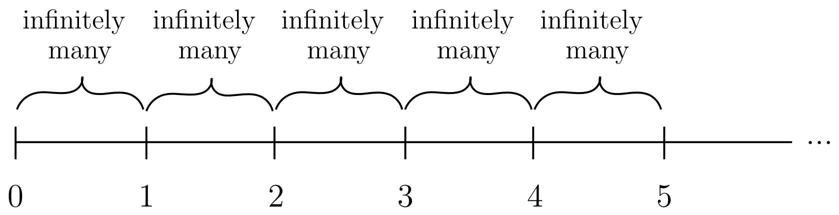

Fig. 3.19

At first glance, trying to prove the equality $|\mathbb{N}| = |\mathbb{Q}^+|$ does not seem very promising. The natural numbers $0, 1, 2, 3, \ldots$ lie very thinly on the right half of the axes, and between any two consecutive natural numbers i and $i+1$ there are infinitely many rational numbers. Additionally, we know that a matching between $\mathbb{N}$ and $\mathbb{Q}^+$ would provide a numbering of elements in $\mathbb{Q}^+$. What does such a numbering of positive rational numbers look like? It cannot follow the size of the rational numbers, because, as we know, there is no smallest positive rational number[11].

Though this very clear impression, we show that the equality

[11] For any small rational number a, one can get the smaller rational number $a/2$ by halving a.

$$|\mathbb{N}| = |\mathbb{Q}^+|$$

and so, in some sense that

$$\infty \cdot \infty = \infty$$

holds.

Observe first that the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ also does not have any smallest number, and though we can number their elements as follows:

$$0, -1, 1, -2, 2, -3, 3, \ldots \ .$$

The idea for $\mathbb{Q}^+$ is to write all positive rational numbers on an infinite sheet as follows (mathematicians among us would say that one assigns positions of the two-dimensional infinite matrix to positive rational numbers). Each positive rational number can be written as

$$\frac{p}{q} \ ,$$

where p and q are positive integers. We partition the infinite sheet of paper into infinitely many columns and infinitely many rows. We number the rows by

$$1, 2, 3, 4, 5, \ldots$$

from top to bottom, and we number the columns from left to right (Fig. 3.20). We place the fraction

$$\frac{i}{j}$$

on the square in which the i-th row intersects the j-th column. In this way we get the infinite matrix as described in Fig. 3.20.

We do not have any doubt that this infinite sheet (this infinite matrix) contains all positive fractions. If one looks for an arbitrary fraction p/q, one immediately knows that p/q is placed on the intersection of the p-th row and the q-th column. But we have

	1	2	3	4	5	6	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	...
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.20

another problem. Some[12] positive rational numbers occur in the matrix several times, even infinitely many times. For instance, the number 1 can be represented as a fraction in the following different ways:

$$\frac{1}{1}\,,\ \frac{2}{2}\,,\ \frac{3}{3}\,,\ \frac{4}{4}\,,\ \ldots\,.$$

The rational number $1/2$ can be written as

$$\frac{1}{2}\,,\ \frac{2}{4}\,,\ \frac{3}{6}\,,\ \frac{4}{8}\,,\ \ldots\,.$$

Exercise 3.13 Which infinitely many representations as a fraction does the rational number $\frac{3}{7}$ have?

But we aim to have each positive rational number appearing exactly once on this sheet. Therefore, we take the fraction p/q that cannot be reduced[13] as a unique representation of the rational number p/q. In this way 1 uniquely represents $1/1$, one half is represented by $1/2$, because all other fractions represented by 1 and

[12] in fact all

[13] The greatest common divisor of p and q is 1.

1/2 can be reduced. Hence, we remove (rub out) all fractions of the sheet that can be reduced. In this way we get empty positions (squares) on the intersections of some rows and columns, but this does not disturb us.

	1	2	3	4	5	6	$\dots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\dots$
2	$\frac{2}{1}$		$\frac{2}{3}$		$\frac{2}{5}$		$\dots$
3	$\frac{3}{1}$	$\frac{3}{2}$		$\frac{3}{4}$	$\frac{3}{5}$		$\dots$
4	$\frac{4}{1}$		$\frac{4}{3}$		$\frac{4}{5}$		$\dots$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$		$\frac{5}{6}$	$\dots$
6	$\frac{6}{1}$				$\frac{6}{5}$		$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.21

Now we want to number the fractions in Fig. 3.21 as the first, the second, the third, etc. Clearly, we cannot do it in the way in which first the elements (fractions) of the first row are numbered, then the elements of the second row, etc., since the number of elements in the first row is infinite. We would fail in such an attempt because we could never start to number the elements of the second row. The first row would simply consume all numbers of $\mathbb{N}$. Analogously, it is impossible to number the elements of the infinite sheet column by column. What can we do then? We number the elements of the sheet in Fig. 3.21 diagonal by diagonal. The ***k*-th diagonal of the sheet** contains all positions (Fig. 3.22) for which the sum of its row number i and its column number j is $k+1$ ($i+j=k+1$).

In this way the first diagonal contains only one element, $\frac{1}{1}$. The second diagonal contains two elements, $\frac{2}{1}$ and $\frac{1}{2}$. And, for instance,

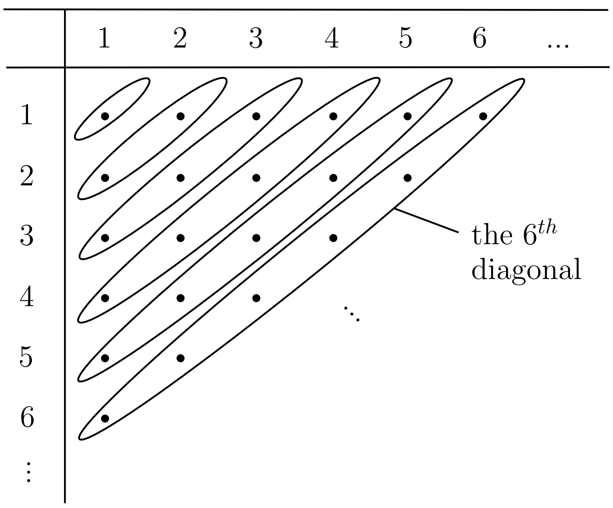

Fig. 3.22

the fourth diagonal contains the four elements $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}$, and $\frac{1}{4}$. In general, for each positive integer k, the k-th diagonal contains exactly k positions, and so at most k fractions.

Now, we order (number) the positions of the infinite sheet, and in this way we order the fractions laying there as shown in Fig. 3.23.

We order the diagonals according to their numbers, and we order the elements of any diagonal from the left to the right. Following this strategy and the placement of the fractions in Fig. 3.21, we obtain the following numbering of all positive rational numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{1}{5}, \frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \ldots.$$

Following our numbering convention, 1/1 is the 0-th rational number, 2/1 is the first positive rational number, etc. For instance, 3/1 is the third rational number, and 5/2 is the 12-th one.

Exercise 3.14 Extend the matrix in Fig. 3.21 by two more rows and columns and place the corresponding fractions in their visible positions. Use this extended

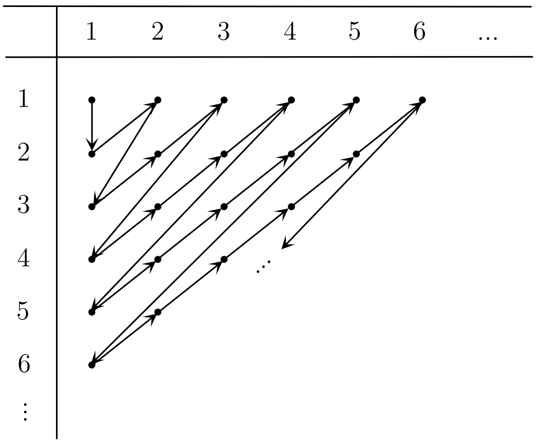

Fig. 3.23

matrix to write the sequence of fractions that got the orders $17, 18, 19, \ldots, 26, 27$ by our numbering.

The most important observation for seeing the correctness of our numbering strategy is that each positive rational number (fraction) is assigned a natural number as its order. The argument is straightforward. Let p/q be an arbitrary positive fraction. The rational number p/q is placed on the intersection of the p-th row and the q-th column, and so it lies on the diagonal $(p+q-1)$. Because **each diagonal contains finitely many positions (fractions)**, the numbering of elements of the forthcoming diagonals $1, 2, 3, \ldots, p+q-2$ is completed in a finite time, and so the numbering of the elements of the diagonal $p+q-1$ is performed too. In this way, p/q as an element of the diagonal $p+q-1$ is also given an order. Since the i-th diagonal contains at most i rational numbers, the order of p/q is at most

$$1 + 2 + 3 + 4 + \ldots + (p + q - 1) \ .$$

In this way, one can conclude that

$$|\mathbb{Q}^+| = |\mathbb{N}|$$

holds.

Exercise 3.15 Figure 3.24 shows another strategy for numbering of positive rational numbers that is also based on the consecutive numbering of diagonals. Write the first 20 positive rational numbers with respect to this numbering. What order is assigned to the fraction 7/3? What order does the number 7/3 have in our original numbering following the numbering strategy depicted in Fig. 3.23?

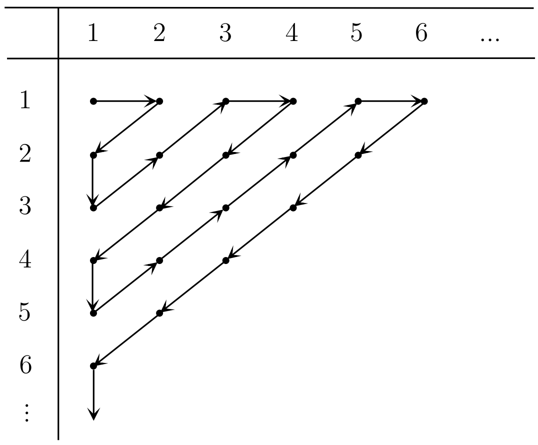

Fig. 3.24

Exercise 3.16 Hotel Hilbert is completely empty; that is no guest is staying there. At once (as it sometimes happens in real life), infinitely many infinite buses arrive. The buses are numbered as

$$B_0, B_1, B_2, B_3, \ldots\ ,$$

i.e., there are as many buses as $|\mathbb{N}|$. For each $i \in \mathbb{N}$, bus B_i contains infinitely many seats

$$B_i(0), B_i(1), B_i(2), B_1(3), \ldots\ .$$

Each seat is occupied by exactly one passenger. How can the hotel porter accommodate all the passengers?

Exercise 3.17 (challenge) Prove that $|\mathbb{Q}| = |\mathbb{N}|$.

Exercise 3.18 (challenge) We define

$$\mathbb{N}^3 = \{(i, j, k) \ \mid \ i, j, k \in \mathbb{N}\}$$

as the set of all triples (i, j, k) of natural numbers. One can place any natural number on each of the three positions of a triple. Hence, one could say that $|\mathbb{N}^3| = |\mathbb{N}| \cdot |\mathbb{N}| \cdot |\mathbb{N}| = \infty \cdot \infty \cdot \infty = \infty^3$. Show that $|\mathbb{N}^3| = |\mathbb{N}|$, and so that $\infty = \infty^3$ holds.

3.3 There are Different Infinite Sizes, or Why There Are More Real Numbers Than Natural Ones

In Section 3.2 we learned Cantor's concept for comparing the cardinalities of sets. Surprisingly, we discovered that the property distinguishing infinite objects from finite ones is that infinite objects contain proper parts that are as large as the whole. We were unsuccessful in searching for an infinity that is larger than $|\mathbb{N}| = \infty$. Even the unexpected equality $|\mathbb{Q}^+| = |\mathbb{N}|$ holds. This is true even though the rational numbers are infinitely more densely placed on the real axis than the natural ones. This means that $\infty \cdot \infty = \infty$. For each positive integer i, one can even prove that the infinite number

$$\underbrace{|\mathbb{N}| \cdot |\mathbb{N}| \cdot \ldots \cdot |\mathbb{N}|}_{k\ times} = \underbrace{\infty \cdot \infty \cdot \ldots \cdot \infty}_{k\ times} = \infty^k$$

is again the same as $|\mathbb{N}| = \infty$.

We are not far from believing that all infinite sets are of the same size. The next surprise is that the contrary is true. In what follows we show that

$$|\mathbb{R}^+| > |\mathbb{N}| \ .$$

Before reading Section 3.2 one would probably believe that the number of real numbers is greater than the number of natural numbers. But now we know that $|\mathbb{Q}^+| = |\mathbb{N}|$ holds. And the real numbers have similar properties to those of the rational numbers. There is no smallest positive real number, and there are infinitely many real numbers on the real axis between any two different real numbers. Since $|\mathbb{N}| = |\mathbb{Q}^+|$, the inequality $|\mathbb{R}^+| > |\mathbb{N}|$ would directly imply

$$|\mathbb{R}^+| > |\mathbb{Q}^+| \ .$$

Is this not surprising? Later in Chapter 4, we will get a deeper understanding of the difference between the sets $\mathbb{R}$ and $\mathbb{Q}$ that is also responsible for the truthfulness of $|\mathbb{R}| > |\mathbb{Q}|$. For now, we reveal only the idea that, in contrast to real numbers, all rational numbers have a finite representation as fractions. Most of the real numbers do not posses any finite description. In order to prove $|\mathbb{R}^+| > |\mathbb{N}|$, we prove a stronger result. Let $[0, 1]$ be the set of all real numbers between 0 and 1, the numbers 0 and 1 included. We show

$$|[0, 1]| \neq |\mathbb{N}| \ .$$

How can one prove inequality between the cardinalities (sizes) of two infinite sets? For proving equality, one has to find a matching between the two sets considered. This can be complicated, but in some sense it is easy because this is constructive. You find a matching and the work is done. To prove $|A| \neq |B|$ you have to prove that **there does not exist any matching between *A* and *B***.The problem is that there may exist infinitely many strategies for constructing a matching between A and B. How can you exclude the success of any of theses strategies? You cannot check all these infinitely many approaches one after another. When one has to show that something does not exist, then we speak about **proofs of nonexistence**.

> *To prove the nonexistence of an object or the impossibility of an event is the hardest task one can pose to a researcher in natural sciences.*

The word "impossible" is almost forbidden in this context and if one uses it, then we have to be careful of its exact interpretation. A known physician told me that it is possible to reconstruct the original egg from an egg fried in the pan. All is based on the reversibility of physical processes[14] and he was even able to calculate the probability of success for the attempt of creating the original egg. The probability was so small that one could consider the success as a real miracle, but it was greater than 0. There are many things considered to be impossible, though they are possible.

In mathematics we work in an artificial world; because of that we are able to create many proofs of nonexistence of mathematical objects. What remains is the fact that proofs of nonexistence belong to the hardest argumentations in mathematics.

[14]as formulated by quantum mechanics

Let us try to prove that it is impossible to number all real numbers from the interval $[0, 1]$, and so that $|[0, 1]| \neq |\mathbb{N}|$. As already mentioned, we do it by indirect argumentation. We assume that there is a numbering of real numbers from $[0, 1]$, and then we show that this assumption leads to a contradiction, i.e., that a consequence of this assumption is an evident nonsense[15].

If there is a numbering of real numbers in $[0, 1]$ (i.e., if there is a matching between $[0, 1]$ and $\mathbb{N}$), then one can make a list of all real numbers from $[0, 1]$ in a table as shown in Fig. 3.25.

	0	1	2	3	4	...	i	...
1	0.	$\boxed{a_{11}}$	a_{12}	a_{13}	a_{14}	...	a_{1i}	...
2	0.	a_{21}	$\boxed{a_{22}}$	a_{23}	a_{24}	...	a_{2i}	...
3	0.	a_{31}	a_{32}	$\boxed{a_{33}}$	a_{34}	...	a_{3i}	...
4	0.	a_{41}	a_{42}	a_{43}	$\boxed{a_{44}}$	...	a_{4i}	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
i	0.	a_{i1}	a_{i2}	a_{i3}	a_{i4}	...	a_{ii}	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.25

This means that the first number in the list is

$$0.a_{11}a_{12}a_{13}a_{14}\dots\ .$$

[15] Here, we recommend revisiting the schema of indirect proofs presented in Chapter 1. If a consequence of an assertion Z is nonsense or contradicts something known, then the indirect proof schema says that Z does not hold, i.e., that the contrary of Z holds. The contrary of the existence of a matching between $[0, 1]$ and $\mathbb{N}$ is the nonexistence of any matching between $[0, 1]$ and $\mathbb{N}$.

The symbols $a_{11}, a_{12}, a_{13}, \ldots$ are digits. In this representation, a_{11} is the first digit to the right of the decimal point, a_{12} is the second digit, a_{13} is the third one, etc. In general

$$0.a_{i1}a_{i2}a_{i3}a_{i4}\ldots$$

is the i-th real number from $[0, 1]$ in our list (numbering). Our table is infinite in both directions. The number of rows is $|\mathbb{N}|$ and the number of columns is also $|\mathbb{N}|$, where the j-th column contains j-th digits behind the decimal points of all numbered real numbers in the list. The number of columns must be infinite, because most real numbers cannot be represented exactly by a bounded number of decimal positions behind the decimal point. For instance, the representation of the fraction

$$\frac{1}{3} = 0.\overline{3} = 0.33333\ldots$$

requires infinitely many digits to the right of the decimal point. On the other hand, this real number is nice because it is periodic. Numbers such as $\sqrt{2}/2$ and $\pi/4$ are not periodic and require infinitely many positions behind the decimal point for their decimal representation.

To be more transparent, we depict a concrete fraction of a hypothetical list of all real numbers from $[0, 1]$ in Fig. 3.26 by exchanging the abstract symbols a_{ij} for concrete digits.

In this hypothetical list the number $0.732110\ldots$ is the first real number, $0.000000\ldots$ is the second real number, etc.

In what follows, we apply the so-called **diagonalization method** in order to show that there is a real number from $[0, 1]$ missing in the list (Fig. 3.25). This contradicts our assumption that one has a numbering of the elements of $[0, 1]$ (i.e., each number from $[0, 1]$ has to occur in the list exactly once). Hence, our hypothetical numbering is not a numbering, and we are allowed to conclude that there does not exist any numbering of the elements from $[0, 1]$.

Next, we construct a number c from $[0, 1]$ that is not represented by any row of the table (list), i.e., that differs from all numbers of the list. We create c digit by digit. We write c as

	0	1	2	3	4	5	6	...
1	0.	7	3	2	1	1	0	...
2	0.	0	0	0	0	0	0	...
3	0.	9	9	8	1	0	3	...
4	0.	2	3	4	0	7	8	...
5	0.	3	5	0	1	1	2	...
6	0.	3	1	4	0	5	7	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
i	0.	7	6	5	0	0	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 3.26

$$c = 0.c_1c_2c_3c_4 \ldots c_i \ldots \ ,$$

i.e., c_i is the i-th digit of c behind the decimal point. We choose $c_1 = a_{11} - 1$ if $a_{11} \neq 0$, and we set $c_1 = 1$ if $a_{11} = 0$. For the hypothetical numbering in Fig. 3.26 this means that $c_1 = 6$ because $a_{11} = 7$. Now we know with certainty that c is different from the number written in the first row of our list in Fig. 3.25 (Fig. 3.26). The second digit c_2 of c is again chosen in such a way that it differs from a_{22}. We take $c_2 = a_{22} - 1$ if $a_{22} \neq 0$, and we set $c_2 = 1$ if $a_{22} = 0$. Hence, c differs from the number in the second row of the list, and so c is not the second number of the hypothetical numbering. Next, one chooses c_3 in such a way that $c_3 \neq a_{33}$ in order to assure that c is not represented by the third row of the list.

In general, one chooses $c_i = a_{ii} - 1$ for $a_{ii} \neq 0$, and $c_i = 1$ for $a_{11} = 0$. In this way c differs from the i-th number of our hypothetical numbering. After six construction steps for the table in Fig. 3.26 one gets

$$0.617106\ldots\ .$$

We immediately see that c differs from the numbers in the first 6 rows of the table in Fig. 3.26.

We observe that c differs from each number of the list in at least one decimal digit, and so c is not in the list. Therefore, the table in Fig. 3.26 is not a numbering of $[0,1]$. A numbering of $[0,1]$ has to list all real numbers from $[0,1]$, and c is clearly in $[0,1]$. Hence, our assumption that one has a numbering of $[0,1]$ (that there exists a numbering of $[0,1]$) is false. We are allowed to conclude

> *There does not exist any numbering of* $[0,1]$*, and so there is no matching between* $\mathbb{N}$ *and* $[0,1]$

Exercise 3.19 Draw a table (as we did in Fig. 3.26) of a hypothetical numbering of $[0,1]$ that starts with the numbers $1/4, 1/8, \sqrt{2}/2, 0, 1, \pi/4, 3/7$. Use this table to determine the digits $c_1, c_2, \ldots, c_7$ of the number c in such a way that c differs from the numbers in the first seven rows of your table.

Exercise 3.20 Consider a hypothetical numbering of $[0,1]$, such that the 100-th number is $2/3$. Which digit of c is determined by this information?

Exercise 3.21 Determine the first seven digits of c behind the decimal point of a hypothetical numbering of $[0,1]$ presented in Fig. 3.27.

What exactly did we show and what was our argumentation? Assume somebody says, "I have a numbering of $[0,1]$." We discovered a method, called diagonalization, that enables us to reject any proposal of a numbering of $[0,1]$ as incomplete because at least one number from $[0,1]$ is missing there. Since we can do it for each hypothetical numbering of the elements of $[0,1]$, there does not exist any (complete) numbering of $[0,1]$.

Another point of view is that of indirect argumentation introduced in Chapter 1. Our aim was to prove the claim Z that there does not exist any numbering of $[0,1]$. We start with the opposite claim $\overline{Z}$ and show that a consequence of $\overline{Z}$ is a nonsense. In this moment we reached our goal. The assertion $\overline{Z}$ as the opposite of Z is the claim that there exists a numbering of the elements of $[0,1]$.

	0	1	2	3	4	5	6	7	...
1	0.	**2**	0	0	1	7	8	0	...
2	0.	1	**7**	3	1	7	8	4	...
3	0.	1	6	**4**	3	3	3	3	...
4	0.	1	6	3	**0**	7	8	4	...
5	0.	1	6	3	1	**8**	8	4	...
6	0.	1	6	3	1	7	**9**	4	...
7	0.	1	6	3	1	7	8	**4**	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 3.27

Starting from $\overline{Z}$ we show that in any such numbering of $[0,1]$ one number from $[0,1]$ is missing. This is nonsense because no number is allowed to be missing in a numbering. Therefore, $\overline{Z}$ does not hold, and so there does not exist any numbering of $[0,1]$.

Since we cannot number the elements of $[0,1]$ (there is no matching between $\mathbb{N}$ and $[0,1]$), we cannot number the elements of $\mathbb{R}^+$, either.

Exercise 3.22 Explain why the nonexistence of a numbering of the elements of $[0,1]$ implies the nonexistence of a numbering of the elements of $\mathbb{R}^+$.

Hint: You can try to explain how to transform each numbering of $\mathbb{R}^+$ into a numbering of $[0,1]$. Why is this a correct argument?

Since $\mathbb{N} \subset \mathbb{R}^+$ and there is no matching between $\mathbb{N}$ and $\mathbb{R}^+$, we can conclude that

$$|\mathbb{N}| < |\mathbb{R}^+|$$

holds. Hence, there are at least two infinite sets of different sizes, namely $\mathbb{N}$ and $\mathbb{R}^+$. One can even show that there are unboundedly many (infinitely many) different infinite sizes. We omit to deal with the technical proof of this result here because we do not need it for reaching our main goal. We are ready to show in the next chapter that the number of computing tasks is larger than the number of algorithms, and so that there exist problems that cannot be solved algorithmically (automatically by the means of computers).

Exercise 3.23 Let us change the diagonalization method presented in Fig. 3.25 a little bit. For each $i \in \mathbb{N}$, we choose $c_i = a_{i,2i} - 1$ for $a_{i,2i} \neq 0$ and $c_i = 1$ for $a_{i,2i} = 0$.

a) Are we allowed again to say that the number $0.c_1c_2c_3c_4\ldots$ is not included in the list? Argue for your answer!
b) Frame the digits $a_{i,2i}$ of the table in Fig. 3.25.
c) Which values are assigned to c_1, c_2, and c_3 for the hypothetic list in Fig. 3.27 in this way? Explain why the created number $c = 0.c_1c_2c_3\ldots$ is not among the first three numbers of the table.

3.4 The Most Important Ideas Once Again

Two infinite sizes can be compared. One has to represent them by the cardinalities of the two sets. Using this as a basis, Cantor introduced the concept for comparing infinite sizes (cardinalities) of two sets by the shepherd's principle. Two sets are equally sized if one can match their elements. A set A has the same cardinality as $\mathbb{N}$ if one can number all elements of A by natural numbers. Clearly, each numbering of A corresponds to a matching between A and $\mathbb{N}$. Surprisingly, one can match $\mathbb{N}$ and $\mathbb{Z}$, though $\mathbb{N}$ is a proper part of $\mathbb{Z}$. In this way we recognized that the property

having a proper part that is as large as the whole

is exactly the characteristic that enables us to distinguish finite objects from infinite ones. No finite object may have this property. For infinite objects, this is a must. Though there are infinitely many rational numbers between any two consecutive natural numbers i and $i + 1$, we found a clever enumeration[16] of all positive rational numbers, and so we showed that $|\mathbb{N}| = |\mathbb{Q}^+|$. After that,

[16] not according to their sizes

we applied the schema of indirect proofs in order to show that there is no numbering of all positive real numbers, and so that there is no matching between $\mathbb{N}$ and $\mathbb{R}^+$.

In Chapter 4, it remains to show that the number of programs is equal to $|\mathbb{N}|$, and that the number of algorithmic tasks is at least $|\mathbb{R}^+|$.

In Chapter 3, we did not present any miracle of computer science. But we did investigate the nature of infinity and the concept of comparing infinite sizes, and in this way we learned miracles of mathematics that are real jewels of the fundamentals of science. Jewels are not found lying on the street, and one usually has to do something to obtain them. Therefore, we are also required to sweat a bit in order to grasp infinity. And so, one may not be surprised that taking our path to the computer science miracles can be strenuous. But tenacity is a good property and the aim is worth the effort. Let us stay this course in the next two chapters, and then we will witness one miracle after the other. We will experience unexpected and elegant solutions to hopeless situations that increase the pulse of each friend of science. Only by patience and hard work, can one attain knowledge that is really valuable.

Solutions to Some Exercises

Exercise 3.1 For the sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$ there are $4! = 24$ different matchings. For instance,

$$(2, 11), (3, 2), (4, 5), (5, 7)$$

or

$$(2, 11), (3, 7), (4, 5), (5, 2) \ .$$

The sequence of pairs $(2, 2), (4, 5), (5, 11), (2, 7)$ is not a matching between A and B because element 2 of A occurs in two pairs, $(2, 2)$ and $(2, 7)$, and element 3 of A does not occur in any pair.

Exercise 3.8 A matching between $\mathbb{N}$ and $\mathbb{Z}$ can be found in such a way that one orders the elements of $\mathbb{Z}$ in the following sequence

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots, i, -i, \ldots$$

and then creates a matching by assigning to each element of $\mathbb{Z}$ its order in this sequence. In this way we get the matching

$$(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3),\ldots\ .$$

In general we build the pairs

$$(0,0),(2i,-i) \text{ and } (2i-1,i)$$

for all positive integers i.

Exercise 3.10 (challenge) First, the porter partitions all rooms into infinitely many groups, each of an infinite size. Always when a group of guests arrives (it does not matter whether the group is finite or infinite), the porter accommodates the guest in the next (still unused) group of rooms.

As usual for the staff of Hotel Hilbert, the porter is well educated in mathematics, and so he knows that there are infinitely many primes

$$2,3,5,7,11,13,17,19,\ldots\ .$$

Let p_i be the i-th prime of this sequence. The porter uses p_i to determine the i-th infinite group of natural numbers as follows:

$$\text{group}(i) = \{p_i, p_i^2, p_i^3, p_i^4, \ldots, (p_i)^j, \ldots\}\ .$$

For instance, group(2)= $\{3,9,27,81,\ldots\}$. Due to his knowledge of the fundamental theorem of arithmetics, the porter knows that no natural number belongs to more than one group. Using this partition of rooms into the groups with respect to their room numbers, the porter can assign the rooms to the guests without any more rooms even when infinitely many groups of guests arrive one after each other. It does not matter, whether the i-th group of guest is finite or infinite, the porter books the whole room group group(i) for the i-th guest group. If the guests of the i-th group are denoted as

$$G_{i,1}, G_{i,2}, G_{i,3}, \ldots, G_{i,j}, \ldots$$

then guest $G_{i,1}$ gets the room $Z(p_i)$, guest $G_{i,2}$ gets room $Z(p_i^2)$, etc.

Exercise 3.12 The sequence of pairs

$$(0,0),(1,1),(2,4),(3,9),(4,16),\ldots,(i,i^2),\ldots$$

is a matching between $\mathbb{N}$ and $\mathbb{N}_{quad}$. We see that each number from $\mathbb{N}$ appears exactly once as the first element in a pair, and analogously each integer from $\mathbb{N}_{quad}$ can be found exactly once as the second element of a pair.

Exercise 3.20 The decimal representation of the fraction 2/3 is

$$0.\overline{6} = 0.666666\ldots\ .$$

Hence, the 100-th position behind the decimal point is also 6. Therefore, one sets $c_{100} = 6-1 = 5$.

Exercise 3.21 For the hypothetical numbering of real numbers from interval $[0,1]$ in Fig. 3.27, one gets

$$c = 0.1631783\ldots\ .$$

Exercise 3.22 We perform an indirect proof by following the schema of the indirect argumentation from Chapter 1. We know that there is no numbering of $[0, 1]$. The aim is to show that there does not exist any numbering of $\mathbb{R}^+$. Assume the contrary of our aim, i.e., that there is a numbering of $\mathbb{R}^+$. We consider this numbering of $\mathbb{R}^+$ as a list and erase those numbers of this list that are not from $[0, 1]$. What remains is the list of numbers from $[0, 1]$ that is (without any doubts) a numbering of $[0, 1]$. But we know that there does not exist any numbering of $[0, 1]$, and so the contrary of our assumption must hold. The contrary of our assumption is our aim, i.e., that there does not exist any numbering of $\mathbb{R}^+$.