D-INFK
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Linear Algebra

HS 2023

Assignment 1

Submission Deadline: **03 October, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. **Rank and linear independence (hand-in)** (★☆☆)

   a) Are the following three vectors in $\mathbb{R}^3$ linearly independent?

$$\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

   b) Are the following three vectors in $\mathbb{R}^4$ linearly independent?

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

   c) What is the rank of the following 2×3 matrix A?

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & 6 & 0 \end{bmatrix}$$

   d) What is the rank of the following 3×3 matrix A? You may use the (yet unproven) statement from the lecture that says that one can choose any order on the columns of a matrix to compute its rank.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

2. **Scalar product** (★★☆)
   Recall that the scalar product of two vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

   in $\mathbb{R}^n$ is a real number given by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n .$$

   and that vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

a) Let $A \in \mathbb{R}^{m \times n}$ be the matrix

$$A = \begin{bmatrix} - & \mathbf{u}_1 & - \\ - & \mathbf{u}_2 & - \\ & \vdots & \\ - & \mathbf{u}_m & - \end{bmatrix}$$

with rows $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathbb{R}^n$. Prove that we have $A\mathbf{x} = \mathbf{0}$ (with $\mathbf{x} \in \mathbb{R}^n$) if and only if $\mathbf{x}$ is perpendicular to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

b) Now consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ and let $c, d \in \mathbb{R}$ be arbitrary. Prove that the vector $c\mathbf{x} + d\mathbf{y}$ is perpendicular to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

c) Finally, consider the set of vectors $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ and assume $|\mathcal{L}| \geq 2$. Is $\mathcal{L}$ a finite set?

3. Equal matrices (★☆☆)

Recall that we use a_{ij} to denote the entry in the i-th row and j-th column of a matrix A. We can use this notation to define a matrix where each entry is a function of its indices. For example, the notation

$$A = (a_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } a_{ij} = i \cdot j$$

reads as: A is a 3×3 matrix with entry $a_{ij} = i \cdot j$ for all $1 \leq i \leq 3$ and $1 \leq j \leq 3$. More concretely, we have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Which of the following matrices are the same?

$$\begin{aligned}
A &= (a_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } a_{ij} = i \cdot j \\
B &= (b_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } b_{ij} = i + j \\
C &= (c_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } c_{ij} = \max\{i, j\} \\
D &= (d_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } d_{ij} = \frac{i+j}{2} + \frac{|i-j|}{2} \\
E &= (e_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } e_{ij} = i \cdot \left(\frac{j^2 - 1}{j+1} + 1 \right) \\
F &= (f_{ij}) \in \mathbb{R}^{3 \times 3} \text{ for } f_{ij} = (i+1) + (j-1)
\end{aligned}$$

4. Lines in $\mathbb{R}^n$ (★★☆)

a) Let $\mathbf{0} \in \mathbb{R}^n$ denote the vector whose entries are all zero. We say that a set L is a line in $\mathbb{R}^n$ if and only if there exists $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{w} \neq \mathbf{0}$ such that $L = \{c\mathbf{w} : c \in \mathbb{R}\}$. Let now L be a line in $\mathbb{R}^n$ and let $\mathbf{u}$ be an arbitrary non-zero element of L. Prove that $L = \{c\mathbf{u} : c \in \mathbb{R}\}$.

b) For two lines L_1 and L_2 in $\mathbb{R}^n$, prove that we have either $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

c) Consider a line L in $\mathbb{R}^2$. Prove that L is a hyperplane, i.e. find a vector $\mathbf{d} \neq \mathbf{0}$ such that

$$L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}.$$

5. Angle between two vectors (★★★)

Consider two non-zero vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$ in $\mathbb{R}^3$ with $x + y + z = 0$. Determine the value of $\cos(\alpha)$ where α denotes the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. You are not required to compute (or look up) α, but you are of course welcome to do so.