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# Assignment 10

Submission Deadline: **05 December, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

## Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

**1. Computing determinants (hand-in)** (★☆☆)

**a)** For what values of $a, b, c \in \mathbb{R}$ is the determinant of the following matrix zero? (You should justify your answer.)

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 & c \\ a & 5 & 0 & 4 & -1 \\ 2 & 1 & b & -1 & -3 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 3 & 1 \end{bmatrix}$$

***Hint**: Use Proposition 5.1.13.*

**b)** We know that the determinant of a triangular matrix is easy to calculate. Moreover, the determinant does not change when a multiple of a row is added to another row (and row swaps only change the sign). This allows us to efficiently determine the determinant of any matrix using Gauss elimination (or $LU$-decompositions).
Determine the determinant of

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

by performing the Gauss elimination manually.

**2. Determinant of block matrix** (★★☆)

**a)** Consider the four matrices

$$\begin{aligned} A &\in \mathbb{R}^{m \times m} \\ C &\in \mathbb{R}^{(n-m)\times(n-m)} \\ B &\in \mathbb{R}^{m \times (n-m)} \\ 0 &\in \mathbb{R}^{(n-m)\times m}. \end{aligned}$$

where $m, n \in \mathbb{N}^+$ with $n > m$. We can plug these matrices together as follows

$$M := \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

to obtain the $n \times n$ matrix $M$. Prove that we have $\det M = (\det A)(\det C)$.

***Hint**: Use Definition 5.1.6.*

**b)** Calculate the determinant of the following matrix

$$M = \begin{bmatrix} 2 & 0 & 0 & 4 & 0 & 7 \\ 9 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 5 & 0 & 7 \\ 2 & 3 & 1 & 5 & 0 & 2 \\ 8 & 8 & 7 & 3 & 2 & 1 \end{bmatrix}$$

by hand without using Gauss elimination.

***Hint**: Put $M$ into the correct form and use the result from the previous subtask. Proposition 5.1.9 and Proposition 5.1.18 might be helpful.*

**3. Linear transformations** (★☆☆)

This task includes Challenge 28, Challenge 29, parts of Challenge 30, and Challenge 34.

**a)** Let $T : U \to V$ be a linear transformation between vector spaces $U$ and $V$. Prove that we have

$$T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k) = \alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_k T(\mathbf{u}_k)$$

for all $k \in \mathbb{N}^+$, all $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$, and all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.

***Hint**: Use induction over $k$.*

**b)** Let $U$ and $V$ be vector spaces, let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in U$ be a basis of $U$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ be arbitrary. Prove that there is a linear transformation $T : U \to V$ with $T(\mathbf{u}_i) = \mathbf{v}_i$ for all $i \in [n]$.

**c)** Let $T : \mathbb{R}^n \to \mathbb{R}$ be the function given by $T(\mathbf{x}) = \| \mathbf{x} \|$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $T$ is not a linear transformation.

**d)** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $L : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations such that $T(\mathbf{x}) = A\mathbf{x}$ and $L(\mathbf{y}) = B\mathbf{y}$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $\mathbf{y} \in \mathbb{R}^m$. Prove that the linear transformation $L \circ T$ (given by $L \circ T(\mathbf{x}) = L(T(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$) satisfies $L \circ T(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

**4. Inverse and determinant** (★☆☆)

This task includes Challenge 36 from the lecture notes.

Consider an arbitrary invertible $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a, b, c, d \in \mathbb{R}$.

**a)** Compute the four co-factors $C_{11}, C_{12}, C_{21}, C_{22}$ of $A$.

**b)** Let $C$ be the matrix that contains those co-factors, i.e.

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

Compute $\frac{1}{\det(A)} C^\top$ in terms of $a, b, c, d$.

**5. Skew-symmetric matrices** (★★☆)

A square matrix $A$ is called skew-symmetric if and only if $A = -A^\top$.

**a)** Calculate the determinant of the skew-symmetric matrix $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ for arbitrary $a \in \mathbb{R}$.

**b)** Calculate the determinant of the skew-symmetric matrix $B = \begin{bmatrix} 0 & b & -1 \\ -b & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ for arbitrary $b \in \mathbb{R}$.

**c)** Prove that for any $n \times n$ matrix $A$, we have $\det(-A) = (-1)^n \det A$.

*Hint: Use Proposition 5.1.19.*

**d)** Let $C \in \mathbb{R}^{n \times n}$ be an arbitrary skew-symmetric matrix where $n \in \mathbb{N}^+$ is odd. What can you say about the determinant of $C$?

**6. Application: as-smooth-as-possible discrete curves (★★★)**

Note that this question is concerned with a nice application of Linear Algebra in Visual Computing. It is not necessarily tied to this weeks topic of determinants. Also, the last subtask involves using a computer to find the solution to a particular example. We recommend solving this exercise last.

That being said, in this question, we wish to fit a smooth-looking curve to a set of points in the plane. For this, we first need to introduce the notion of closed discrete curves.

A *closed discrete curve* in the plane is an ordered list of points $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ with $\mathbf{p}_j \in \mathbb{R}^2$ for all $j \in [n]$. The points are called *vertices* of the curve, and we consider each pair of consecutive vertices on the curve to be connected by a straight line segment (this also includes a straight line segment between $\mathbf{p}_n$ and $\mathbf{p}_1$).

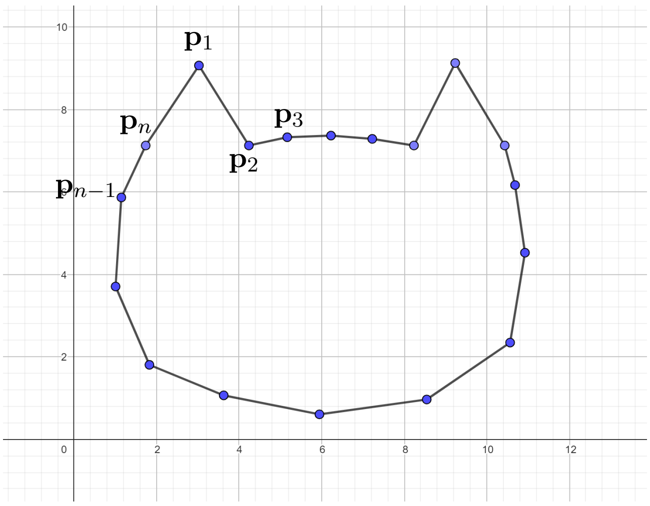


Figure 1: An example of a closed discrete curve.

In this exercise, we want to find a closed discrete curve $P = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ consisting of $n \in \mathbb{N}^+$ points. More concretely, each point $\mathbf{p}_j = \begin{bmatrix} p_{x,j} & p_{y,j} \end{bmatrix}^\top \in \mathbb{R}^2$ (with $j \in [n]$) has $x$-coordinate $p_{x,j}$ and $y$-coordinate $p_{y,j}$. We want to determine those coordinates.

As input, we are given a set of $k \in \mathbb{N}^+$ distinct indices $\mathcal{C} = \{j_1, j_2, \ldots, j_k\}$ that indicate marked vertices on the curve $P$, and a set of corresponding locations in the plane $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ with $\mathbf{c}_s = \begin{bmatrix} c_{x,s} & c_{y,s} \end{bmatrix}^\top \in \mathbb{R}^2$ for all $s \in [k]$.

The task is to compute the locations of all the curve vertices $\mathbf{p}_1, \ldots, \mathbf{p}_n$ such that the marked vertices are as close as possible to the given locations, and such that each curve vertex is as smooth as possible. In equations, ideally we would want to have

$$\mathbf{p}_{j_s} = \mathbf{c}_s, \ \forall\, s \in [k], \tag{1}$$

and

$$\mathbf{p}_j = \frac{1}{2}(\mathbf{p}_{j-1} + \mathbf{p}_{j+1}), \ \ \forall j \in \{2, \ldots, n-1\} \tag{2}$$
$$\mathbf{p}_1 = \frac{1}{2}(\mathbf{p}_n + \mathbf{p}_2), \tag{3}$$
$$\mathbf{p}_n = \frac{1}{2}(\mathbf{p}_{n-1} + \mathbf{p}_1). \tag{4}$$

In particular, equations (2)-(4) aim to capture the notion of smoothness.

**a)** The equations above can be written down as two systems of linear equations, one with unknowns $p_{x,1}, p_{x,2}, \ldots, p_{x,n}$ and the other with unknowns $p_{y,1}, p_{y,2}, \ldots, p_{y,n}$. For example, the constraints in (1) can be separately written as $p_{x,j_s} = c_{x,s}$ and $p_{y,j_s} = c_{y,s}$ for all $s \in [k]$. Similarly, the equations (2)-(4) can be written separately for the $x$- and $y$-components. Sketch the two linear systems in matrix form. In particular, for each linear system determine the system matrix and the right-hand side.

**b)** As it turns out, both systems have the same system matrix. What can you say about the rank of this matrix in terms of $n$ and $k$?

**c)** Consider the following values: $n = 6$, $k = 3$, $\mathcal{C} = \{j_1 = 1, j_2 = 3, j_3 = 5\}$ and

$$\mathbf{c}_1 = \begin{bmatrix} c_{x,1} \\ c_{y,1} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_2 = \begin{bmatrix} c_{x,2} \\ c_{y,2} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_3 = \begin{bmatrix} c_{x,3} \\ c_{y,3} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Combine the two linear systems into one big system with unknowns $p_{x,1}, p_{x,2}, \ldots, p_{x,n}$ and $p_{y,1}, p_{y,2}, \ldots, p_{y,n}$. For the values provided above, solve this system in the least squares sense (you may use the help of a computer) and sketch the solution in the grid below.

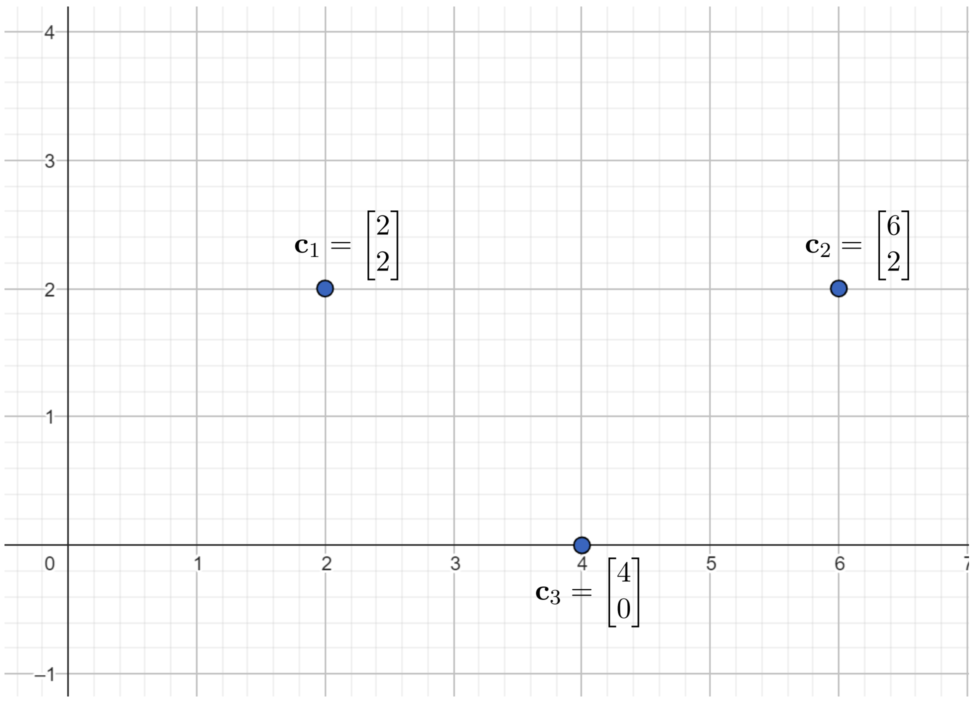


Figure 2: Solve for an as-smooth-as-possible discrete curve with 6 points $\mathbf{p}_1, \ldots, \mathbf{p}_6$ where the vertices $\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_5$ are constrained to be as close as possible to the illustrated positions $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, respectively.