D-INFK | Linear Algebra | HS 2023
Afonso Bandeira
Bernd Gärtner

Assignment 11

Submission Deadline: **12 December, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Eigenvalues of rotations and reflections (★★☆)

Consider two linear transformations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ where Φ is a rotation around the y-axis (i.e. the axis given by $\mathbf{e}_2$) by $\frac{\pi}{3}$, and Ψ is the reflection through the plane $P = \{\begin{bmatrix} x & y & z \end{bmatrix}^\top \in \mathbb{R}^3 : 3x + 4y = 0\}$.

Note that the rotation is specified according to the right-hand rule in a right-handed coordinate system[1]: if you imagine your right thumb to be the vector $\mathbf{e}_2$, then slightly curling the other fingers on your right hand will give you the direction of the rotation. For this to uniquely describe the rotation, it is important to specify that we are thinking of a right-handed coordinate system.

a) Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that $A\mathbf{x} = \Phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

b) Find a matrix $B \in \mathbb{R}^{3\times 3}$ such that $B\mathbf{x} = \Psi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

c) Prove that both A and B are orthogonal.

d) Find one real eigenvalue and a corresponding real eigenvector of A.

Hint: You might not have to calculate them. It's valid to guess them and verify that they are indeed an eigenvalue-eigenvector pair.

e) Find two distinct real eigenvalues of B, and a corresponding real eigenvector for each of them.

Hint: You might not have to calculate them. It's valid to guess them and verify that they are indeed eigenvalue-eigenvector pairs.

2. Challenge 49 (★☆☆)

Challenge 49 asks you to check that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \frac{1+\sqrt{5}}{2} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \frac{1-\sqrt{5}}{2} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

In particular, please check that $1 + \frac{1+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$ and $1 + \frac{1-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$.

[1] https://en.wikipedia.org/wiki/Right-hand_rule

3. Complex numbers (★☆☆)

a) Given the complex numbers

$$u = 3 - i^3,$$
$$v = 1 + i,$$
$$w = 3 - 4i,$$

calculate the expressions $u + v + w$, $u \cdot v$, $v \cdot w \cdot i$, ${}^{w}/_{v}$, ${}^{v}/_{u}$, $|\, v \,|$.

b) Write the complex numbers 3, $2i$, $1 + \sqrt{3}i$ and $5\sqrt{3} - 5i$ in the polar form $re^{i\theta}$.

c) Write the complex numbers $-2e^{i\pi/4}$, $4e^{i2\pi/3}$ in the cartesian form.

4. Drawing complex numbers (★★☆)

Draw the following sets of complex numbers in the complex plane:

a) $A := \{z \in \mathbb{C} : 1 < \Im\,(i\bar{z}) < 2\}$;

b) $B := \{z \in \mathbb{C} : |\, z - 2 \,| < |\, z - 2i \,|\}$;

c) $C := \{z \in \mathbb{C} : (\bar{z} + 1)(z + 1) = 2\Im(z)\}$.

5. Orthonormal eigenvectors (★★★)

Consider the three orthonormal vectors

$$\mathbf{v}_1 = \frac{1}{9}\begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{9}\begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \frac{1}{9}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

with corresponding scalars $\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0.$

a) Let $A \in \mathbb{R}^{3\times 3}$ be a matrix that has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e. $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all $i \in [3]$. What is the value of $\det(A)$?

b) Prove that the vector $\begin{bmatrix} 16 & 2 & 8 \end{bmatrix}^\top$ belongs to the nullspace of A.

c) Find such a matrix A.

6. Eigenvalues when adding cI to matrices (★★☆)

a) Let $M \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$. Show that for each real eigenvalue $\lambda \in \mathbb{R}$ of M, $\lambda + c$ is a real eigenvalue of $M + cI$.

b) Using the property from **a)**, find two distinct real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ of the matrix

$$A = \begin{bmatrix} 3 & 3 & 5 & 7 & 9 & 11 \\ 1 & 5 & 5 & 7 & 9 & 11 \\ 1 & 3 & 7 & 7 & 9 & 11 \\ 1 & 3 & 5 & 9 & 9 & 11 \\ 1 & 3 & 5 & 7 & 11 & 11 \\ 1 & 3 & 5 & 7 & 9 & 13 \end{bmatrix}.$$

c) Determine the dimensions of the subspaces $\mathbf{N}(A - \lambda_1 I)$ and $\mathbf{N}(A - \lambda_2 I)$, respectively.

7. Roots of complex numbers (★★☆)

a) Find all $z \in \mathbb{C}$ that satisfy $3z^3 + 81 = 0$.

b) Find all $z \in \mathbb{C}$ that satisfy $2z^2 + 4i = 0$.

c) Find all $z \in \mathbb{C}$ that satisfy $z^2 - \sqrt{2}(2 - i^3 + e^{i\pi}) = 0$.

8. Recursively defined sequences (★★☆)

Consider the sequence of numbers given by $a_0 = 1, a_1 = 1$ and $a_n = 3a_{n-1} + 4a_{n-2}$ for $n \geq 2$. Find a closed-form formula for a_n in terms of $n \in \mathbb{N}$.