D-INFK
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Linear Algebra

HS 2023

Assignment 12

Submission Deadline: **19 December, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Eigenvalues and eigenvectors of AB and BA (hand-in) (★★☆)

Let A, B be two matrices in $\mathbb{R}^{n\times n}$.

a) Let $\lambda \in \mathbb{R}$ be a real eigenvalue of AB. Prove that λ is a real eigenvalue of BA.

b) Assume that B is invertible and that AB has a complete set of real eigenvectors (according to Definition 6.1.18). Prove that BA has a complete set of real eigenvectors.

c) Assume that both A and B are invertible. Prove that AB has a complete set of real eigenvectors if and only if BA has a complete set of real eigenvectors.

d) Can you find an example of two matrices A and B such that BA has a complete set of real eigenvectors, but AB does not have a complete set of real eigenvectors?

2. Diagonalization and similarity (★★☆)

This exercise includes Challenge 62 and Challenge 63.

Let A, B, C be matrices in $\mathbb{R}^{n\times n}$.

a) Assume that A has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Assume further that B has the same eigenvectors as A (i.e. every eigenvector of A is an eigenvector of B and vice versa). Prove that $AB = BA$.

b) Assume that A and B are similar. Further assume that B and C are similar. Prove that A and C are similar.

c) Assume that A and B have the same n distinct real eigenvalues. Prove that A and B are similar.

d) Assume that A and B are similar. Prove that they have the same real eigenvalues. Note that you are not allowed to use Proposition 6.2.4 for this exercise.

3. Eigenvalues (★★☆)

a) Let $A \in \mathbb{R}^{n\times n}$ be an arbitrary matrix and consider two linearly independent vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $A\mathbf{v} = 3\mathbf{w}$ and $A\mathbf{w} = 3\mathbf{v}$. Prove that both 3 and -3 are eigenvalues of A.

b) Let $A \in \mathbb{R}^{n\times n}$ be an arbitrary matrix and consider two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} \neq \mathbf{0}$, $A\mathbf{v} = 3\mathbf{w}$, and $A\mathbf{w} = 3\mathbf{v}$. Prove that 3 or -3 is an eigenvalue of A.

4. Application: population with three age groups (★★☆)

Consider a species with three age groups. Let x_t denote the number of individuals in the first age group at time $t \in \mathbb{N}_0$, y_t the number of individuals in the second age group at time $t \in \mathbb{N}_0$, and z_t the number of individuals in the third age group at time $t \in \mathbb{N}_0$. Moreover, assume the following:

- At every point in time, half of the individuals from the first age group make it to the second age group (the other half dies).
- At every point in time, a third of the individuals from the second age group make it to the third age group (the other two thirds die).
- At every point in time, each individual in the second group leads to one newborn individual (in group one).
- At every point in time, each individual in the third group leads to three newborns (in group one).

For $t \in \mathbb{N}_0$, we use the vector $\mathbf{v}_t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^\top$ to summarize the current population. Hence, according to the description above, we have $\mathbf{v}_{t+1} = A\mathbf{v}_t$ where

$$A = \begin{bmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Is there a choice for the initial population $\mathbf{v}_0 \in \mathbb{R}^3$ such that the population will be stable over time (i.e. $\mathbf{v}_t = \mathbf{v}_{t+1}$ for all $t \in \mathbb{N}_0$)? If possible, make sure that all entries of $\mathbf{v}_0$ are non-negative since that seems more realistic.

5. Eigenvalues and eigenvectors (★☆☆)

a) Let $A \in \mathbb{R}^{2\times 2}$ be such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ for all $x, y \in \mathbb{R}$. Find all real eigenvalues of A. For each real eigenvalue, find a corresponding real eigenvector of A.

b) Construct a square matrix A with eigenvalues $0, 1, 2$. Furthermore, these should be the only eigenvalues of A.

c) Construct a square matrix B with eigenvalues $0, 1, 2$ such that B is not a diagonal matrix. As before, these should be the only eigenvalues of B.