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Assignment 13

Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

There will be no hand-in for this assignment. Solutions will be published on December 22.

Exercises

1. Positive (semi-)definite matrices (★★☆)

This exercise includes Challenge 68 which asks you to prove Fact 6.3.13. Note that you already saw the proof of Fact 6.3.13 and the solution to Challenge 68 in the lecture. But you can still use this exercise to see if you can repicate the proof yourself.

Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Let $\lambda_{\text{min}}^{(A)} \in \mathbb{R}$ be the smallest eigenvalue of A, let $\lambda_{\text{min}}^{(B)} \in \mathbb{R}$ be the smallest eigenvalue of B, and let $\lambda_{\text{min}}^{(A+B)} \in \mathbb{R}$ be the smallest eigenvalue of $A+B$.

a) Prove that $\lambda_{\text{min}}^{(A+B)} \geq \lambda_{\text{min}}^{(A)} + \lambda_{\text{min}}^{(B)}$.

b) Assume that both A and B are positive semidefinite. Prove that $A+B$ is positive semidefinite.

c) Assume that both A and B are positive definite. Prove that $A+B$ is positive definite.

2. Pseudoinverse via SVD (★★☆)

This exercise includes parts of Challenge 70.

Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank r with singular value decomposition (SVD) $A = U_r \Sigma_r V_r^\top$ with $U_r \in \mathbb{R}^{m\times r}$, $\Sigma_r \in \mathbb{R}^{r\times r}$, and $V_r \in \mathbb{R}^{n\times r}$. Recall that A has a pseudoinverse $A^\dagger$. Note that Σ_r is invertible since it is a square diagonal matrix with non-zero entries on its diagonal. Prove that $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$.

3. Least squares via SVD (★★☆)

This exercise includes parts of Challenge 70.

In this task, we derive the solution of the least squares method using the singular value decomposition. Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$ and $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. Let $A = U\Sigma V^\top$ be an SVD of A with $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, and $V \in \mathbb{R}^{n\times n}$. Consider the least squares problem

$$\arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2. \tag{1}$$

a) Let $\mathbf{c} = U^\top \mathbf{b}$. Prove that $\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2 = \min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$.

b) Let $\sigma_1 \geq \cdots \geq \sigma_r$ denote the non-zero singular values of A (r is the rank of A). In particular, we have $\Sigma_{ii} = \sigma_i$ for all $i \in [r]$. Find a formula for the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ in terms of $\sigma_1, \dots, \sigma_r$ and $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_m \end{bmatrix}^\top$.

c) Let $\mathbf{x}^*$ be the optimal solution $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2$. Given the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ and the SVD of A, how can you compute $\mathbf{x}^*$?

4. Euclidean norm and 1-norm (Manhattan distance) (★★★)

This exercise includes Challenge 71. You can find the relevant definitions in Section 7.2 of the lecture notes. While Section 7.2 was not covered in the lecture, we still decided to include this exercise here since it does not require much extra theory (only the definition of the 1-norm) and we think it is a nice exercise.

a) Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Prove that $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$.

Hint: Try to first prove it for $n = 2$. It might help to square both sides of the inequality first.

b) Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Prove that $\|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2$.

Hint: Recall the Cauchy-Schwarz inequality and try to find a nice way to write $\|\mathbf{x}\|_1$ as the dot product of two vectors.

5. Matrix norms (★★★)

This exercise includes Challenge 72 that asks you to prove Proposition 7.2.2. You can find the relevant definitions in Section 7.2 of the lecture notes. While Section 7.2 was not covered in the lecture, we still decided to include this exercise here since it does not require much extra theory (only the definition of the Frobenius norm and the operator norm) and we think it is a nice exercise.

Let $A \in \mathbb{R}^{m\times n}$ be arbitrary and let $\sigma_1 \geq \cdots \geq \sigma_{\min\{m,n\}}$ be its singular values.

a) Prove that $\|A\|_F^2 = \mathrm{Tr}(A^\top A)$.

b) Prove that $\|A\|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2$.

c) Prove that $\|A\|_{op} = \sigma_1$.

d) Prove that $\|A\|_{op} \leq \|A\|_F$.

e) Prove that $\|A\|_F \leq \sqrt{\min\{m,n\}}\,\|A\|_{op}$.

Remark: Compare d) and e) to Exercise 4.