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Assignment 2

Submission Deadline: **10 October, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Matrix multiplication (hand-in) (★★☆)

a) For a natural number $k \geq 1$, we define the k-th power of a square matrix A as $A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$ where $\times$ denotes matrix multiplication. Moroever, we define $A^0 = I$.

Now consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $x, y, z \in \mathbb{R}$ such that $A^3 + xA^2 + yA + zI = 0$. Note that both I and 0 are 3×3 matrices in this equation.

b) Let A and B be $n \times n$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

c) We say that a square matrix A is nilpotent if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the nilpotent degree of A.

Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

d) Let A be an $n \times n$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \ldots + A^{k-1}) = I$.

e) Let T be an $n \times n$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^n = 0$, i.e. T is nilpotent of degree less or equal to n.

Hint: Even if you do not manage to solve a question, you can use its result to tackle subsequent questions.

2. Solving linear systems (★☆☆) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most 2, i.e. $p(x) = ax^2 + bx + c$ for some coefficients $a, b, c \in \mathbb{R}$. Assume that we already know $p(-1) = 0$, $p(0) = 2$ and $p(1) = 2$. Find the coefficients a, b and c. As the title suggests, you will have to solve a linear system. We recommend that you do it by using the systematic elimination procedure from the lecture.

3. Rank-1 matrices (★☆☆)

a) Consider the 3×3 matrix

$$A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$$

with $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$. What can you say about the rank of A? How does this change if we additionally assume $v_1 \neq 0$ and $w_1 \neq 0$?

b) Assume now $v_1 \neq 0$ and $w_1 \neq 0$ and consider the equation $A\mathbf{x} = \mathbf{0}$ with $\mathbf{x} \in \mathbb{R}^3$. Provide a non-zero solution for $\mathbf{x}$ in terms of $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$ (non-zero means that it cannot be the zero-vector $\mathbf{0}$).

c) Consider the set of solutions $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$ to the above equation. Prove that $\mathcal{L}$ is a hyperplane of $\mathbb{R}^3$.

4. Rotation matrices (★★☆)

***Hint**: This exercise requires some basic knowledge of* $\sin$ *and* $\cos$*. Part c) can also be solved independently by assuming parts a) and b).*

a) A real 2×2 matrix A is a *rotation matrix* if there exists a rotation angle $\phi \in \mathbb{R}$ such that

$$A = Q(\phi) := \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Prove that the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a rotation matrix.

b) Show that the matrix product $Q(\phi_1)\,Q(\phi_2)$ of two rotation matrices with angles ϕ_1 and ϕ_2 is again a rotation matrix $Q(\phi_3)$ and determine the corresponding rotation angle ϕ_3.

***Hint**: You might need to review trigonometric formulas to solve this question.*

c) Let A be a 2×2 rotation matrix. Prove that there exists a 2×2 matrix B such that $AB = BA = I$.

5. Multiple choice

Let A be an $m_1 \times n_1$ matrix and let B be an $m_2 \times n_2$ matrix for natural numbers m_1, n_1, m_2, n_2. For each statement, determine whether it is true or not (regardless of what values m_1, n_1, m_2, n_2 take).

1. If A^2 is defined, then A must be square.

(a) Yes

(b) No

2. If $A^2 = I$, then $A = I$.

(a) Yes

(b) No

3. If $A^3 = 0$, then $A = 0$.

(**a**) Yes

(**b**) No

4. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$ for all $n \in \mathbb{N}$.

(**a**) Yes

(**b**) No

5. If $AB = B$ for some choice of B, then $A = I$.

(**a**) Yes

(**b**) No

6. If both products AB and BA are defined, then A and B must be square.

(**a**) Yes

(**b**) No

7. If both products AB and BA are defined, then AB and BA must be square.

(**a**) Yes

(**b**) No

8. If two columns of A are equal and AB is defined, the corresponding columns of AB must also be equal.

(**a**) Yes

(**b**) No

9. If two columns of B are equal and AB is defined, the corresponding columns of AB must also be equal.

(**a**) Yes

(**b**) No

10. If two rows of A are equal and AB is defined, the corresponding rows of AB must also be equal.

(**a**) Yes

(**b**) No

11. If two rows of B are equal and AB is defined, the corresponding rows of AB must also be equal.

(**a**) Yes

(**b**) No

12. If A and B are symmetric matrices and AB is defined, AB is also symmetric.

(**a**) Yes

(**b**) No